PROVING ACTIVITIES IN INQUIRIES USING THE INTERNET

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This paper elucidates the nature of proving activities required in the inquiry-based learning of mathematics using the Internet, wherein the didactic contract is different from that in the ordinary mathematics classroom. Based on the anthropological theory of didactics, proving activities conducted in the study and research paths are explored in the context of Japanese pre-service mathematics teacher education. We design and implement situations for finding the cube root of a given number by using a simple pocket calculator. The analysis of the realised situations shows that inquiries using the Internet generate, in a way adidactic, students' different activities related to the proof, such as reading proofs, posing new why-questions, proving by themselves to understand the information obtained on the Internet and the method of calculation.

INTRODUCTION

The difficulties of learning proof and proving are well known, and this has been the subject of a significant body of research (cf. Mariotti, 2006). One difficulty which is often discussed, especially in the authors' country, is the necessity of proofs (MEXT, 2009). Students do not feel the necessity of proving a statement, particularly statements already known as true since elementary school (e.g. properties of a parallelogram). However, teachers also face difficulties in creating learning situations in which proofs are required to solve a problem, that is, situations wherein students feel the necessity of proving. We consider that this difficulty is, to some extent, due to the *didactic contract* (Brousseau, 1997) which is created in ordinary teaching and learning situations in mathematics classrooms, and due to the *paramathematical* nature of a proof (Chevallard, 1985/1991): proof is a tool for studying mathematics rather than a mathematical object to be studied (except in mathematical logic). Since a proof is a paramathematical object, its teaching cannot be dissociated from other mathematical knowledge to be taught. In the classroom, what is justified by the proof is the statement related to this knowledge, and this statement to be proven is always true because what is taught in school is the set of organized objects which are known to be true. There is a contract, that the teacher teaches or education generally provides 'true' knowledge to students. Students know that the statement to be proven is true before proving it, since it is given by the teacher and it is a piece of knowledge which students have to know.

What if the didactic contract differs from that found in the ordinary mathematics classroom? What kinds of proving activities would be conducted? Further, is it possible to radically change such a didactic contract? In a recent study, a 'new' way to conceive mathematics teaching is proposed, and the didactic contract created in such teaching seems very different from the ordinary didactic contract. It is a sequence of activities called *Study and Research Paths* (SRP hereafter: Chevallard, 2006; 2015) within the *Anthropological Theory of the Didactic* (ATD) developed by Chevallard. SRP is based on the didactic paradigm called *questioning the world* (Chevallard, 2015), in which the learning is aimed at nurturing scientists' attitudes in the process of elaborating an answer to a question. Students investigate a question by means of any tool available (e.g. calculator, computer, Internet, any books), and mathematical knowledge is learnt through a process when necessary. Unlike teaching based on the 'old' paradigm wherein *raison d'être* or rationale as to why students should learn it is often implicit, mathematical knowledge to be taught is not organized in a sequence to be learnt one-by-one, and it is accompanied by a *raison d'être.* Additionally, it might be the teacher who proposes the initial question, but there is no specific expected answer and no specific mathematical knowledge to be taught. The teacher's role is that of a supervisor of scientific research. The didactic contract is thus very different from the ordinary mathematics classroom.

In such inquiries, what kinds of proving activities would be required and conducted especially in the case of inquiries using the Internet? We investigate this question by designing and implementing situations based on the idea of SRP in the context of Japanese pre-service mathematics teacher education. Through an analysis of the realised situations, we try to identify the nature of proving activities in such situations. We expect that different activities related to the proof, difficult to conduct in ordinary teaching, will be identified while the students elaborate an answer to the question.

THEORETICAL FRAMEWORK

In what follows, we briefly introduce the notion of SRP, which plays a crucial role in this study. It is used as a conceptual tool to develop the learning situations to be realised and as an analytical tool to clarify the nature of students' activities conducted in the situations realised in the teaching experiment. In ATD, inquiries in mathematics and other fields are characterised by the notion of SRP (cf. Barquero & Bosch, 2015). SRP expresses dialectic processes between questions and answers, where an inquirer starts from an initial question Q_0 and arrives at a final answer $A^\blacktriangledown$. The simplest SRP is modelled as '$Q_0 \rightarrow A^\blacktriangledown$'. However, the process of finding an answer includes other steps. The inquirer usually encounters another various questions Q_k derived from the initial question or others, and finds answers A_k to them. Some answers could have already been produced by the predecessors: those are labelled as $A_i^\lozenge$. This process is modelled, for example, as $Q_0 \rightarrow Q_1 \rightarrow A_1 \rightarrow Q_2 \rightarrow A_2^\lozenge \rightarrow Q_3 \rightarrow A^\blacktriangledown$. However, most study processes cannot be formulated by a linear diagram but by a tree diagram, because a question often leads to multiple questions.

Further, the process of the elaboration of an answer is characterised in ATD by the media-milieu dialectic. Similar to its use in the Theory of Didactic Situations (TDS), a milieu refers to a system without didactic intention, acting as a fragment of 'nature', with which the inquirer interacts during the study process (cf. Chevallard, 2004; Artigue et al., 2010; Kidron et al., 2014). In contrast, the media refers to any system with the intention of supplying information about the world or a part of it to a certain type of audience (cf. Chevallard, 2004; Artigue et al., 2010; Kidron et al., 2014). In order to get an answer to a question, the inquirer looks for and obtains information from media, and elaborates an answer by interacting with the milieu including such information. SRP based on the *questioning the world* presupposes the use of media as in scientists' activities, restricted in ordinary teaching based on the 'old' paradigm.

METHODOLOGY

In this study, we design and implement learning situations based on the idea of SRP and analyse the data collected in the experiment in order to clarify the nature of proving activities in inquiries. We adopt as a methodology *didactic engineering within ATD*, which includes four phases of the analysis and design of didactic phenomena: preliminary analysis; conception and *a priori* analysis; experimentation and *in vivo* analysis; and *a posteriori* analysis (cf. Barquero & Bosch, 2015). In this paper, we report some parts of these analyses.

As we mentioned above, the notion of SRP is used as a conceptual tool to design learning situations. It allows us not only to design tools for students to use in class (e.g., Internet), but also to consider the nature of the initial question Q_0 proposed to them: Q_0 should be an *alive question*, so that it is connected with various mathematical or other activities; Q_0 should have *generative power*, so that many other questions Q_k are derived. We looked for such an initial question and designed a sequence of situations in the context of pre-service mathematics teacher education. The details of the design are revealed in the next section.

In the experiment, we collected students' worksheets, PC screen views which show the history of pages visited on the Internet, and the video and audio data for the entire lessons and the activities of each group which were translated later. In the analysis, the SRP is now used as an analytical tool. The *tree structure of questions and answers in SRP* allows us to model the dynamics and process of inquiry, and the media-milieu dialectic allows us to model the dynamics of mathematical activities. Specifically, in the *in vivo* analysis, we first identify various questions Q_k posed by students, answers obtained from the media $A_i^{\Diamond}$, and temporary or final answers elaborated A_k or $A^{\heartsuit}$, from which are constructed a diagram representing a tree structure of SRP. Further, we describe, by means of the media-milieu dialectic, students' activities related to these questions and answers, in particular those concerning proving. Then we discuss, as an *a posteriori* analysis, the nature of the proving activities required in SRP, based on the results of the *in vivo* analysis.

MATHEMATICAL AND DIDACTIC DESIGN: *A PRIORI* ANALYSIS

We design situations in the context of pre-service mathematics teacher education in a university dedicated to elementary-school teacher training. Target students are third-year undergraduate students enrolled in a program to obtain a secondary-school mathematics teacher certificate, in addition to the elementary-school teacher certificate. In general, students in this university are not very competent in mathematics.

$A_{0\text{-}1}$	$A_{0\text{-}2}$
1. [a] [√] [√]	1. [a] [√] [√] [×]
2. [×] [a] [=] [√] [√]	2. [a] [√] [√] [√] [√] [×]
3. [×] [a] [=] [√] [√]	3. [a] [√] [√] [√] [√] [√] [√] [×]
4. ...	4. ...

Fig. 1. Two answers to the initial question Q_0. (a is a given number)

The initial question Q_0 we chose is: how to calculate the cube root of a given number by using a simple pocket calculator? The calculator has the function of calculating a square root, in addition to the four basic operations (+, –, × and ÷), but nothing other than these functions. This question is generally well known in Japan, and one may find different websites related to it on the Internet. The question is closed and its answer could be easily found in the media. However, starting from this question, students might ask other different questions that lead to the various mathematical concepts. In this sense, we consider that Q_0 is an *alive question* which has *generative power*.

In search for the answer to Q_0, one may find two methods of calculation $A_{0\text{-}1}$ and $A_{0\text{-}2}$ given in Fig. l. The naïve question derived from these answers, for the students of the university, is the question Q_1: why does such a method allow the calculation of the cube root? The answer to this question A_1 could be found in the media (websites) or through interacting with a milieu. For example, the operations on the calculator could be translated into an infinite series on the exponent part which converges to 1/3 (the operations of $A_{0\text{-}1}$ to the first line of Fig. 2 and the operations of $A_{0\text{-}2}$ to the second line). At this point, students are exposed to mathematical works on infinite series, such as the limit of series and the recurrence relation, and are required to read and understand the proof obtained from the media, which is $A^\lozenge$, or to prove by themselves. Further, the question of calculating the cube root of a given number would also derive questions related to the calculation of the nth root, such as the 5th root and 7th root. Developing an answer to such a question allows students to encounter other mathematical works such as those related to the Mersenne numbers $2^k – 1$ (appearing when solving a recurrence relation such as $x_{n+1} = (x_n\ a^p)$^$(1/2)^q$), binary numbers (converting 1/n to a binary representation provides an infinite series like the second line of Fig. 2), etc.

In the class, students will be asked to conduct the inquiry based on their own interests. While some questions will be provided by the teacher, the derived questions might or might not be the ones we anticipated above. Students deal with the questions they pose on their own. There is no specific mathematical knowledge expected for the students to acquire (open SRP). The objective of the class is to nurture scientists' attitude and to develop students' views on mathematical activities (SRP

$$
\begin{aligned}
&((((((\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}+1)\frac{1}{4}\cdots \\
=\ &\frac{1}{4}+(\frac{1}{4})^2+(\frac{1}{4})^3+(\frac{1}{4})^4+\cdots+(\frac{1}{4})^n+\cdots \\
=\ &\frac{1}{4}\lim_{n\to\infty}\frac{1-(\frac{1}{4})^n}{1-\frac{1}{4}}=\frac{1}{4}\cdot\frac{4}{3}\cdot 1=\frac{1}{3}
\end{aligned}
$$

Fig. 2. The series on the exponents converges to 1/3

for teacher training). In such situations, the didactic contract should be different from that in ordinary situations.

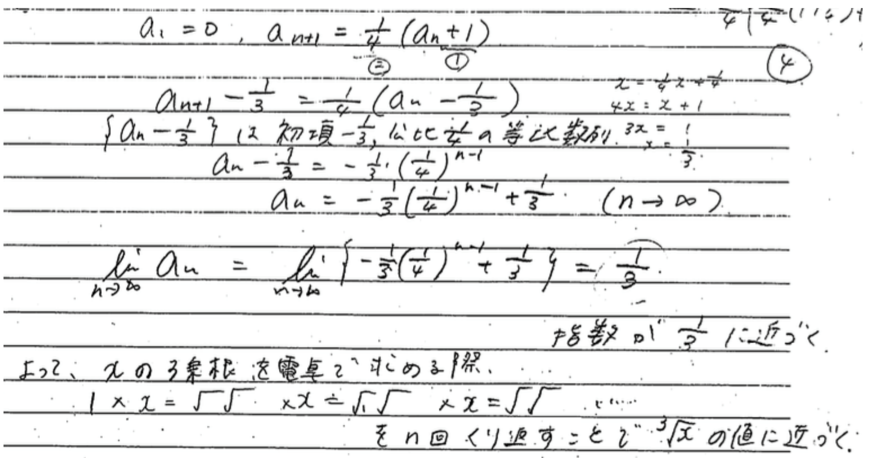
EXPERIMENTATION: *IN VIVO* ANALYSIS

The third author of this paper taught a class based on the situations designed in the *a priori* phase. This class includes three teaching periods of 90 minutes (one period per week), allocated to the inquiries using the Internet, and one period for the presentation of the results of the inquiries. Nine students assisted in this class. The inquiry was conducted in a group of three students. Thus, three groups were created. A pocket calculator was provided to each student, and a laptop PC connected to Wi-Fi was provided to each group. At the beginning of the first period, in addition to providing the initial question Q_0, the teacher explained the objective of the class and the modality of the inquiry. The objective is for students to experience and know the 'authentic' mathematical activities that mathematicians conduct in their research. The students may use any tools (media) to advance their inquiries; there is no final goal expected by the teacher and the inquiries may follow any direction, depending on the students' interests and their new questions. The teacher's role is to support their inquiries. In the last period, they should present the products of their inquiries. At this stage, the teacher tried to devolve the situations so that the students and the teacher could create a didactic contract which is specific to the inquiries.

Overall, each group worked sincerely during the three teaching periods and also during the time-out period of the class, for preparing a presentation. In the first period, the inquiry is conducted especially for identifying the method to calculate the cube root of a given number and to understand why such a method works. From the second to third periods, each group inquires into its own question and proceeds towards different directions: the first group proceeded to the calculation of the *n*th root, the second group to another justification of the calculation method by using a graphic representation of the convergence, and the third group to the speed of convergence.

We describe here the process of inquiry through an analysis of students' activities from the theoretical perspective of ATD, particularly SRP and the media-milieu dialectic. In the *in vivo* and *a posteriori* phases, we focus on SRP of the first period in the second group (Group 2 hereafter). In the beginning of the inquiry for an answer to Q_0, Group 2 immediately reached a first webpage, 'calculation of cube roots using a calculator' (http://www004.upp.so-net.ne.jp/s_honma/urawaza/root.htm). This page introduces a method of calculation by a simple calculator. The explanation starts with the recurrence relation of exponents '$a_1 = a$, $4a_{n+1} = a_n + 1$', and then introduces the method '[a] [×] [N] [=] [√] [√]; [×] [N] [=] [√] [√]; [×] [N] [=] [√] [√] ...' in the case of '$a = 2$, $N = 2$'. The explanation justifying the method is given in a way 'mathematical'. The recurrent relation is given at first without *raison d'être*, and then the formula corresponding to the method ($X_{n+1} = \sqrt{\sqrt{X_n \times N}}$) is deduced. Group 2 firstly regarded the given solution as $A_{0\text{-}1}^{\lozenge}$. This answer generated a new question Q_1: why should we consider '$4a_{n+1} = a_n + 1$'? The students was trying to determine the general term a_n by themselves in interacting with the milieu. However, at that moment, the teacher intervened and asked again Q_0 about the method of calculation. Indeed, the students read and follow the proof for a method, although they did not know the method itself. This teacher's intervention lead the students to focus on the method given in the first website $A_{0\text{-}1}^{\lozenge}$. Further, this information from the media prompted the use of calculators as a part of their milieu. The students worked back and forth between reading the proof on the webpage and calculating using a calculator and found that this method works after checking it with different numbers. The method they verified became their own answer A_0. However, two new questions were produced successively: 'why does such method works?' (Q_2) and 'why could the first number a be arbitrary?' (Q_3). Related to these questions, the small questions and answers could be identified. For example, they asked about the operations of calculator like 'why are there so many repetitions?' In fact, they did not even realised at the first moment that the repetitive operations and its convergent value correspond respectively to the recurrent relation and the limits of a series. After a short moment, they found an answer related to the limit of a series. For Q_3, they asked by themselves the meaning of 'arbitrary' and were searching an answer on the Internet. They found some explanation on the websites, but they understood rather in the second website ($A_{0\text{-}2}^{\lozenge}$) about the method of calculating cube roots, wherein the page explains the same method as that of the first website and writes the first number can be any number such as 1, 2, 3 (http://www.nishnet.ne.jp/~math/mr_boo/DENTAKU1.HTM).

In searching for the answers to Q_2, the students found the third webpage ($A_{0\text{-}3}^{\lozenge}$: http://blog.livedoor.jp/ddrerizayoi/archives/26225078.html). This page provides the same method as before in the case of '$a = 1$, $N = 7$', and also a justification with the recurrence relation of exponents. In contrast to the first and second webpages, the third one explicitly describes the process of exponential changes in each operation: 0 → 1 → 1/4 → (1/4) + 1 → (1/4)((1/4) + 1) .... The students interacted with this information as a part of milieu and advanced their inquiry. They first realized the relationship between the operation on the calculator and the number of exponent and also how the recurrent relation given in $A_{0\text{-}3}^{\lozenge}$ ($a_1 = a$, $a_{n+1} = 1/4(a_n + 1)$) relates to the operations. In


$a_1 = 0$, $a_{n+1} = \frac{1}{4}(a_n + 1)$

$a_{n+1} - \frac{1}{3} = \frac{1}{4}(a_n - \frac{1}{3})$

$\{a_n - \frac{1}{3}\}$ は初項 $-\frac{1}{3}$, 公比 $\frac{1}{4}$ の等比数列

$a_n - \frac{1}{3} = -\frac{1}{3}(\frac{1}{4})^{n-1}$

$a_n = -\frac{1}{3}(\frac{1}{4})^{n-1} + \frac{1}{3}$ $(n \to \infty)$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\{-\frac{1}{3}(\frac{1}{4})^{n-1} + \frac{1}{3}\} = \frac{1}{3}$

指数が $\frac{1}{3}$ に近づく

よって、x の3乗根を電卓で求める際、

$1 \times x = \sqrt{}\sqrt{}$ $\times x = \sqrt{}\sqrt{}$ $\times x = \sqrt{}\sqrt{}$ ...

を n 回くり返すことで $\sqrt[3]{x}$ の値に近づく。

$x = \frac{1}{4}x + \frac{1}{4}$, $4x = x + 1$, $3x = 1$, $x = \frac{1}{3}$

Fig. 3. A_2: the proof written by a member of Group 2

this website, while the limit of the series '1/3' is given, its proof was not given. Reading this page, the students found the general term a_n by themselves give a proof like Fig. 3. This is thus their devised answer A_2 to Q_2.

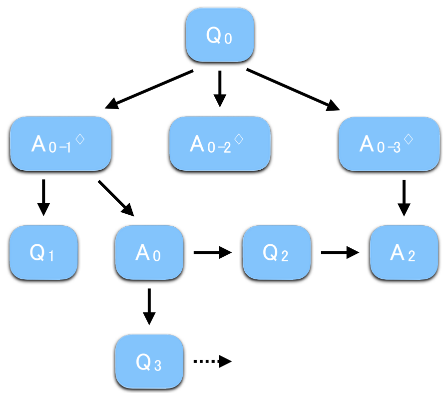


Fig. 4. Tree structure of SRP of Group 2

After getting A_2, the question Q_3 became one of main questions the students of Group 2 tackled in the last part of the first teaching period. We could not provide details here. However, they carried out different activities such as observing the behavior of convergence when changing the initial number a in the spreadsheet. These processes of inquiry are summarised as Fig. 4.

DISCUSSION: *A POSTERIORI* ANALYSIS

In SRP of Fig. 4, three questions Q_1, Q_2 and Q_3 emerged not from the teacher but from the students through the media-milieu dialectics. For example, Q_2 and Q_3 were generated, while they were reading the proof given in the first webpage ($A_{0\text{-}1}^{\lozenge}$), that is to say, Q_2 and Q_3 were produced as a result of the interaction with the milieu including $A_{0\text{-}1}^{\lozenge}$ obtained from the media. What is interesting here is that these questions require a kind of proving activities, while Q_0 asked by the teacher requires just providing a method which could be easily found on the Internet. Further, Q_3 was not expected by the teacher while Q_2 was. In ordinary teaching situations, the question asked by students would not be dealt with as a main issue, because they are based on a didactic contract that the teacher has exclusively legitimacy about questioning (e.g. Chevallard, 2015). In addition, the teacher has a difficulty of creating a situation wherein students ask by themselves why-questions and elaborate their justification to them, as we have discussed earlier. However, in the situations of SRP, such activities could be easily observed.

On the other hand, a written mathematical proof was given only for Q_2, and Q_3 was investigated empirically at least in this teaching period. Nevertheless, the students validated the method A_0 on their own by interacting with their milieu, and made their own answer A_2 to the question Q_2 by proving a statement. In this step, the students constructed a proof in order to understand the method of calculation and the answer $A_{0\text{-}3}^{\lozenge}$ obtained from the media. The proving for understanding is unfortunately infrequent in ordinary class, although Hanna pointed out that 'proof can make its greatest contribution in the classroom only when the teacher is able to use proofs that convey understanding' (2000, p. 7). The mathematical understanding should be a principal role of proof. However, to what extent does the proving activities carried out by secondary students in mathematics classroom really lead the mathematical understanding? We consider that the inquiries using the Internet like the SRP have a possibility for overcoming this problem.

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