# A THEORETICAL FRAMEWORK FOR CURRICULUM DEVELOPMENT IN THE TEACHING OF MATHEMATICAL PROOF AT THE SECONDARY SCHOOL LEVEL

Yusuke Shinno[1], Takeshi Miyakawa[2], Hideki Iwasaki[3],

Susumu Kunimune[4], Tatsuya Mizoguchi[5], Terumasa Ishii[6], Yoshitaka Abe[7]

[1]Osaka Kyoiku University, [2]Joetsu University of Education, [3]Hiroshima University,

[4]Shizuoka University, [5]Tottori University, [6]Kyoto University, [7]Niigata University

*The aim of this paper is to construct a theoretical framework for curriculum development in the teaching of mathematical proof at the secondary school level. To accomplish this aim, we first search for, through the review of related literature, the principal aspects of mathematical proof that should be taken into consideration for the framework. In particular, we consider the idea of "local organization" introduced by Freudenthal (1971) and the idea of "mathematical theorem" proposed by an Italian research group (Mariotti et al., 1997). In terms of these ideas, we then develop a framework for teaching mathematical proof and examine elements of the framework in line with mathematics curricular content in Japan. Examples and implications for curriculum development are also discussed.*

## MATHEMATICAL PROOF IN CURRICULUM

Traditionally, the teaching of mathematical proof was relegated to geometry at the secondary school mathematics level. It might be a well-known fact that the majority of students were unable to construct valid proofs. Currently, however, there seems to be a general trend towards including proof and proving at all levels of school mathematics (e.g., NCTM, 2000). Therefore a number of research studies carried out at all levels of mathematics have been reported the teaching and learning of proof and proving in light of explanation, reasoning, argumentation, and so on (e.g., Mariotti, 2006; Stylianou et al., 2009; Reid & Knipping, 2010; Hanna & de Villiers, 2012). In retrospect, what does such an endeavour mean for improving the teaching of mathematical proof at the secondary school level? We think that it is necessary to consider—from the perspective of important results of earlier research on proof and proving—a more synthesised approach to the mathematical or formal proof in curriculum. We cannot ignore the influences of curricular content and sequencing when we analyse students' constructions of mathematical proofs (Hoyles, 1997). However, because of "the huge variation in when proof in introduced and how it is treated in different countries" (Hoyles, 1997, p. 7), only a few attempts have so far been made at a broader discussion of curricular content and sequencing of mathematical proof that could be explicitly introduced at the secondary level in some countries, including Japan. There is room for argument on this point.

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This paper reports on part of an ongoing research project regarding the developmental study of the teaching of mathematical proof throughout six years (Grades 7-12) of secondary schooling in Japan. It focuses on proposing a theoretical framework for curriculum development in the teaching of mathematical proof. For this reason, we must draw attention to the theoretical perspectives with a few examples, but the discussion of empirical aspects of the framework would take us beyond the scope of this paper. Although the present study is targeting Japanese secondary school mathematics, in developing a framework we attempt to synthesise multiple theoretical perspectives well known within the international mathematics education community in order to enable the framework to be comparable with those in other countries. Thus, the research questions in this paper are as follows: *What kinds of teaching contents should be included in the secondary curriculum for the teaching of mathematical proof?* and *What kinds of evolution should be envisioned in the course of the curriculum?*

## THEORETICAL PERSPECTIVES

### "Proof" and "Demonstration"

What is meant by "mathematical proof"? There is the distinction often made in some countries between "proof" and "demonstration". For example, Balacheff (1987) describes the French distinction between "prevue" and "démonstration" as follows:

> We call proof an explanation accepted by a given community at a given moment… Within the mathematical community only explanations adopting a particular form can be accepted as proofs. They are an organised succession of statements following specified rules: a statement is known to be true or is deduced from those which precede it using a deductive rule taken from a well defined set of rules. We call such proofs "démonstrations". (Balacheff, 1987, p. 148: English translation cited from Reid & Knipping (2010, pp. 32-33))

In Balacheff's sense, "démonstration" in French can be translated as "mathematical proof" in English, and it is distinguished from "proof". Although "most English writers do not use 'proof' and 'mathematical proof' in the same way as Balacheff does" (Reid & Knipping, 2010, p. 33), within the Japanese mathematics education community, we sometimes make a similar distinction between "proof (*shoumei*)" and "mathematical proof *(ronshou*)" (e.g., Hirabayashi, 1991; Japan Society of Mathematical Education, 1966). Thus, in this paper we would like to use the word "mathematical proof" in the special sense of "démonstration" as Balacheff says.

The distinction between proof and mathematical proof implies that these words are often discussed in relation to the statements or theorems to be proven and the system of mathematics in which the proof is carried out. We, therefore, attempt to consider organisation or systematisation of statements as the principal aspects of mathematical proof. In order to do so, the idea of "local organization" (Freudenthal, 1971) and the idea of "mathematical theorem" (Mariotti *et al*., 1997) are taken into account.

## Local Organization

Freudenthal (1971; 1973) proposed the idea of local organization and emphasised the significance of mathematical activities based on the local organization in geometry. Local organization is an important didactic idea proposed as distinguished from the idea of global organization based on the axiomatic system:

> Indeed, a student who never exercised organising a subject matter on local levels will not succeed on the global one. (Frendenthal, 1971, p. 426)
>
> In general, what we do if we create and if we apply mathematics, is an activity of local organization. Beginners in mathematics cannot do even more than that. Every teacher knows that most students can produce and understand only short deduction chains. They cannot grasp long proofs as a whole, and still can they view substantial part of mathematics as a deductive system. (ibid., p. 431)

What Freudenthal means by local organization is shown by this example of the proof of the perpendicular bisectors of a triangle. Consider a question by the teacher: "draw the bisectors of *AB* and *BC*, which intersect at *M*; look where the bisector of *AC* passes". Freudenthal provides the analysis of the following proof:

> The proof rests on the property of the bisector of *XY* being the set of all points equidistant from *X* and *Y*, which may have been recognised by symmetry arguments. *M* is on the bisector of *AB* whence
>
> $MA = MB$ ;
>
> *M* is on the bisector of *BC* where
>
> $MB = MC$
>
> From both follows
>
> $MA = MC,$
>
> whence *M* is on the bisector of *AC*. (Freudenthal, 1971, p. 429)

In his view, students need not be able to prove the equidistance property of the perpendicular bisector, because this property may be, for students who do not have the idea of a relational system, taken for granted, and it "cannot contribute anything to the understanding of the circumcircle theorem" (ibid., p. 430). In line with Freudenthal's idea, Hanna and Jahnke (2002) proposed a distinction between "small theory" and "large theory", and they remarked that "instead of building a large theory (namely, Euclidean geometry) in the course of the curriculum, it seems to be more appropriate to work in several small theories" (p. 3). Here it is important to note that the property taken for granted in the local organization or small theory is consistent with the theorem proven in the global organization or large theory. We think that such a distinction can be one of the principal aspects of teaching mathematical proof that should be taken into consideration when developing a curriculum.

## Mathematical Theorem

In order to elaborate on the relationship between mathematical proof and local organization, we consider another important theoretical perspective—the idea of "mathematical theorem" proposed by the Italian research group (Mariotti et al., 1997;

Mariotti, 2006: Antonini & Mariotti, 2008). According to the characterisation by Mariotti et al. (1997), a mathematical theorem consists of a system of relations between a *statement*, its *proof*, and the *theory* within which the proof make sense. Indeed, in mathematicians' mathematical practice, a mathematical assertion such as a proposition and its validation is always considered in a certain theoretical context such as geometrical, arithmetic, algebraic, and other contexts; "the existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense" (Mariotti et al., 1997, p. 182). We consider that these three elements—*statement*, *proof*, and *theory*—that characterise a mathematical theorem can be principal aspects of teaching mathematical proof that evolve throughout secondary school mathematics. We think that, in particular, the nature of *theory* can be well characterised by the idea of local organization.

## ELEMENTS OF A THEORETICAL FRAMEWORK OF TEACHING MATHEMATICAL PROOF

The methodology we adopt in the present study is that of synthesising the theoretical perspectives mentioned in the previous section and of examining the contents and levels of mathematical proof in terms of "statement", "proof", and "theory" in line with mathematics curricular content in Japan. In this way, we develop a framework for teaching mathematical proof that allows us to design a curriculum.

### Contents of "Statement", "Proof", and "Theory"

We first attempt to identify the *contents* of "statement", "proof", and "theory" respectively. Here we lean on logical points of view to identify the different kinds of "statement" that could be included in secondary mathematics. We think that there are four kinds of propositions: a) singular proposition, b) universal proposition, c) existential proposition, and d) other proposition such as negative proposition. Although these four kinds of propositions are included in both primary and secondary school curriculum in Japan, the distinctions between them—such as distinct universal from existential proposition—are not explicitly taught even at the secondary level.

We next consider the contents of "proof" to be types of proof such as: a) direct proof, b) indirect proof, and c) mathematical induction, which are included in the secondary school curriculum. As far as indirect proof is concerned, it is formally introduced in Grade 10 in Japan, but informally students spontaneously produce indirect argumentation (Anotinini & Mariotti, 2008). Therefore it is necessary to examine how we could deal with indirect proof progressively in the course of the curriculum.

In general, the contents of "theory" are both mathematical theory—Euclidean geometry, number theory, and so on—and the logical inference rules, such as modus ponens, conjunctive inference, and so on. In particular, the latter is referred to "meta-theory" (Antonini & Mariotti, 2008). This distinction also becomes important in discussing secondary school mathematics. Although "mathematical theory" can be explicit teaching content, "meta-theory" remains implicit at the secondary level in

Japan. In order to understand what "meta-theory" is like, let us show a proof by contradiction as an example (see Antonini & Mariotti (2008) for detailed analysis).

*Statements*: Let $a$ and $b$ be two real numbers. If $ab = 0$, then $a = 0$ or $b = 0$.

*Proof*: Assume that $ab = 0$, $a \neq 0$, and $b \neq 0$. One can divide both sides of the equality $ab = 0$ by $a$ and by $b$, obtaining $1 = 0$. It is a contradiction ($1 \neq 0$). Therefore $a = 0$ or $b = 0$.

*Theory*: Properties of equality, real numbers.

*Meta-theory*: Law of excluded middle, law of double negation, modus ponens, etc.

## Levels of "Statement", "Proof", and "Theory"

We then attempt to identify the *levels* of "statement", "proof", and "theory" respectively. As far as levels of "statements" are concerned, there are two different kinds of educational evolution in terms of the setting of a proof. One level is about the *object* that the statement refers to. It seems reasonable to suppose that there are two levels: i) an object of the real world, and ii) an object of the mathematical world. For example, in the beginning stage of learning geometry, if the statement (probably a singular proposition) refers to "a written triangle", the object of investigation is in the real or material world. At a higher stage, if the statement (probably a universal proposition) refers to "any triangle", the object of investigation is in mathematical world. Another evolution is about the *formulation* of the statement, because the same statement is able to have different representations. In the course of curriculum, it seems that there are three levels of formulation of the statement: i) figure, manipulation, and gesture; ii) ordinary language and word; and iii) mathematical word and symbol. In the case of the universal proposition, for example, the statement can be formulated as "the sum of the interior angles of *any* triangle is 180°". This formulation is the second level, although the universal quantifier is not represented as the symbol "" ", which is the third level. In Japanese language, we rarely say "*any* triangle" or "*all* triangles" in a textbook or geometry class. Although the third-level formulation is not dealt with in the current curriculum, we think that the progressive formulation of the statement can be a crucial point of the curriculum development in this research project.

Concerning the levels of "proof", we consider two different kinds of evolution. Since these have been discussed in Balacheff's (1987) categories of proof so far, similar categories can be applied to our framework as levels of "proof"; that is, the *validation* and *formulation* of "proof". Since the same may be said about the formulation levels of the statement, here we just mention validation levels. It is fair to say that there are three levels of validation: i) explanation, ii) mathematical proof, and iii) formal proof. "Explanation" includes a discourse by informal reasoning, such as inductive and abductive reasoning. Although both "mathematical proof" and "formal proof" are considered as *intellectual proof* in Balacheff's sense, "formal proof" is based on *naïve formalist* language such as symbolic logic. And "mathematical proof" that can be an accepted discourse in the mathematicians' community which means a simplified version of "formal proof". For the consideration of a transition from one level to a higher level, well-known Balacheff's subcategories—*naïve empiricism*, *crucial experiment*, *generic example*, and *thought experiment*—may be useful.

We rely on Frendenthal's idea of local organization, or on "small theory" and "large theory" by Hanna & Jahnke (2002), in order to characterise different levels of "theory". By focusing on the *nature* of each system within which the proof is carried out, we propose three levels of "theory" as follows: i) logic of the real world, ii) local theory, and iii) (quasi-) axiomatic theory. The first level is not the main focus of the study in secondary mathematics. If one accepts that a geometric property is to be true by means of physical experiment or measurement based on the real world, it can be interpreted that the nature of "theory" is based on "logic of the real world". The distinction between "local theory" and "(quasi-) axiomatic theory" is rather important in secondary schools. The former can be the main focus of study in lower secondary school. We put the label "quasi-" onto "axiomatic theory", because it is not relevant to deal with a globally organised axiomatic system explicitly in secondary school mathematics. As a result, Table 1 provides a summary of the framework that resulted from considering contents and levels of three elements. Additionally, in the next section, since the transition from "local theory" to "quasi-axiomatic theory" can be a key to the curriculum development in upper secondary school, we attempt to draw a brief sketch of such a crucial transitional aspect by means of a mathematics textbook.

| | ***Statement*** | ***Proof*** | ***Theory*** |
|---|---|---|---|
| **Contents** | a. Singular proposition<br>b. Universal proposition<br>c. Existential proposition<br>d. Others | a. Direct proof<br>b. Indirect proof<br>c. Mathematical induction | a. Normal theory (e.g., algebra, geometry, calculus, etc.)<br>b. Meta-theory (e.g., modus ponens, etc.) |
| **Levels** | **Object**<br>i. An object in the real world<br>ii. An objects in the mathematical world<br>**Formulation**<br>i. Figure, manipulation, gesture<br>ii. Ordinary language, word<br>iii. Mathematical word, symbol | **Validation**<br>i. Explanation<br>ii. Mathematical proof<br>iii. Formal proof<br>**Formulation**<br>i. Figure, manipulation, gesture<br>ii. Ordinary language, word<br>iii. Mathematical word, symbol | **Nature of system**<br>i. Logic of the real world<br>ii. Local theory<br>iii. (Quasi-) axiomatic theory |

Table 1: A framework for curriculum development in the teaching of mathematical proof—contents and levels

## EXAMPLES AND IMPLICATIONS FOR CURRICULM DEVELOPMENTS

Let us consider the introduction of mathematical induction [MI] as an example to illustrate the nature of "local theory" and "quasi-axiomatic theory". MI is a teaching content that is included in the teaching unit of sequence in upper secondary school in Japan. MI as a teaching material is a kind of capstone in this teaching unit, which consists of the following items in a textbook that has been mostly used in an 11th Grade class. In Table 2, there is space only for the items (left side) and some excerpts of concrete statements (right side), though proofs are also described in the textbook.

| | |
|---|---|
| §1. Arithmetic sequence and geometric sequence<br>1.1 Sequence and the general term<br>1.2 Arithmetic sequence<br>1.3 Arithmetic series<br>1.4 Geometric sequence<br>1.4 Geometric series | The sum of $S_n$ of the first n terms of an arithmetic sequence with the first term *a* and common difference *d* is given by the following formula.<br>$S_n = \frac{n}{2}\{2a + (n-1)d\}$ |
| §2. Other kinds of sequence<br>2.1 The sigma notation Σ<br>2.2 Difference of sequence<br>2.3 The sum of various series | By using above formula and given identical equation, the following equations are proven.<br>$1+2+\cdots+n = n(n+1)/2$<br>$1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6$ |
| §3. Mathematical induction<br>3.1 Recurrence relation<br>3.2 Mathematical induction | The following statements are proven by mathematical induction.<br>- The sum of the first *n* positive integers is $n(n+1)/2$<br>- The sum of the first $n^2$ positive integers is $n(n+1)(2n+1)/6$ |

Table 2: Outline of the teaching unit "sequence" and some excerpts from a textbook

On the one hand, the contents of §2 can be seen as proof and proving at the level of "local theory", because the accepted formula (e.g., the sum of $S_n$ of the first *n* terms) and/or given identical equation are deductively used for proving the statements (e.g., $1+2+\cdots+n = n(n+1)/2$). But part of the formula used in the proof has been acquired by a generic pictorial explanation that cannot be accepted as mathematical proof (it may be at the level of "logic of the real world"), and ready-made identical equations (e.g., $k^3-(k-1)^3 = 3k^2-3k+1$) without proof are used for proving the statement (e.g., $1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6$). On the other hand, contents of §3 can be seen as proof and proving at the level of "quasi-axiomatic theory" because the statements (some of them are the same statements proved in §2) are proven by appeal to the Principle of Mathematical Induction (Peano's fifth axiom for the foundation of natural number) that permits the application of "a meta-theory" (i. e., modus ponens, etc.) to establish the truth of the statement about the elements of sets that can be placed in one-to-one correspondence with the set **N** (cf. Tall et al., 2012, p. 39). What does it imply for further developmental research? Although the appeal to Peano's axiom is usually implicit in the proof method of MI, it may be worthwhile at this point to relate to the other aspects such as the formulation of "statement" or the validation of "proof", and to investigate how more-precise mathematical words might affect students' proof and proving at the level of "quasi-axiomatic theory" for the sake of curriculum development.

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