PROOF IN GEOMETRY: A COMPARATIVE ANALYSIS OF FRENCH AND JAPANESE TEXTBOOKS

Takeshi MIYAKAWA

Joetsu University of Education, Japan

This paper reports a part of results of the international comparative study on the nature of proof in lower secondary school geometry. In order to clarify the different natures of proof in different countries, textbooks of two countries, France and Japan, are analysed, from the ecological perspective of the Anthropological theory of the didactic. The results of the analysis show several differences in what is called "proof", in the form of proof, in the interrelations between mathematical objects that the proof creates, and in the functions of proof.

INTRODUCTION AND THEORETICAL FRAMEWORK

Learning and teaching of proof has been a topic of research in mathematics education for a long time, and a lot of research has been conducted so far. Recently Balacheff (2008) points out that the meaning of "mathematical proof" is not necessarily shared today among researchers in mathematics education, albeit prior research. He reports the different researcher's epistemologies in mathematics education. Reid & Knipping (2010, Part 1) also extensively describe the different usages of the terms "proof" and "proving" and different perspectives on their learning and teaching. Besides this diversity, taking a glance at the proof in school mathematics of different countries, one may also find diversity of proof: form of proof (cf. two-column proof in US), properties used in proof (cf. triangle congruency in US and in Japan, but not in France), and functions of proof, etc.

However, from the perspective of the Anthropological Theory of the Didactic (ATD), in particular the ecological perspective (Chevallard, 1994, Artaud, 1998), this diversity of proof in school mathematics could be, in a sense, a natural consequence. According to ATD, in different educational systems or institutions (e.g., French secondary school and Japanese one), the body of mathematical knowledge taught or to be taught would be different. A mathematical object exists there not in isolation, but in relation to other objects, with particular functions. It is like in ecology that a species lives in some places (called *habitats*) of an ecosystem with some functions (called *niches*) in relation to other species. And what allows the object or species to live in such particular places in a particular form is a system of *conditions* and *constraints* involved in an environment and imposed to that object. In the case of proof, while the proof is not really a mathematical object but a *paramathematical object* (Chevallard, 1991, Ch.4) which is an auxiliary object in mathematical practice, I consider that the nature of proof in school mathematics is also formed in the same mechanism: the proof in secondary school mathematics has interrelations with some particular objects in a body of

2012. In Tso, T. Y. (Ed.). *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 225-232. Taipei, Taiwan: PME.

mathematical knowledge formed in a given educational system; some functions are attributed to the proof for the sake of mathematical practice and also for the sake of didactical practice in an educational system; internal and external constraints of an educational system affects the nature of proof; etc. In consequence, the proof taught in different countries may have different nature, and what should be learnt, what is really learnt, the difficulties students hold, teacher's supports, etc. may also differ from country to country.

In the research on mathematics education, one could find some literature that report differences on the nature of proof taught in school mathematics. For example, Knipping (2003) reports that in the case of Pythagorean Theorem, the processes of proving play different functions in German class and in French class of 8th grade. Proving is to make clear the meaning of a theorem in German Class, while it is to explain why in French class. Cabassut (2005) reports that the proof is explicitly an object of teaching in French secondary schools and in German gymnasium, while it is not the case in German Realschule and Hauptschule, and shows that there is a mixture of different types of arguments and of different functions of validation from the perspective of didactic transposition.

The aim of this study is to further develop these prior research and examine, from the ecological perspective of ATD, the different natures of proof that may exist in secondary school mathematics of different countries. The principle research questions are: *What is the thing called 'proof' in school mathematics in a specific country? Why and how is it formed as it is?* Contributions I expect to obtain in this research are twofold. On the one hand, it would propose a coherent view of proof albeit differences. This would maximise the results obtained in prior research on proof. On the other hand, it would propose alternative ecologies of proof along with the conditions to be satisfied for their realisation, which would serve for future curriculum development. In this paper, I report a primary result, in particular a response obtained for the first research question, by means of the analyses of textbooks of two countries, France and Japan. The second question related to the system of conditions and constraints that forms the nature of proof will be studies the next time.

METHODOLOGY

In order to clarify different possibilities of the nature of proof, a comparative study will be carried out in the cases of France and Japan. It is expected from a cross cultural comparative study to make explicit what is implicit or taken for granted in other country. For the sake of a comparative analysis of proof, mathematics textbooks of each country will be analysed as data. From the perspective of ATD, especially in the process of didactic transposition, different mathematics are taken into consideration: scholarly mathematics, mathematics to be taught, and taught mathematics (Chevallard, 1991). In the case of Knipping's study (2003), the proof really taught in the classroom was an object of study. On the other hand, in this paper, the object of study is the proof to be taught that could be identified in French and Japanese textbooks.

Mathematics textbooks to be analysed

France and Japan both adopt a single-track educational system for the lower secondary level, that is, all students go to the same kind of school: four years of *collège* in France and three years of *middle school* in Japan. And in both countries, teaching contents are determined in the national curriculum written by the Ministry of Education. Japanese textbook should be approved by the ministry, while no approval is required for French textbook. For the analysis, a mathematics textbook which is relatively known and shared in each country was chosen: for Japanese textbook, *Atarashii Suugaku [New Mathematics]* series published by Tokyo-Shoseki (I call "*Tokyo-shoseki* series"), and for French textbook, *Triangle* series published by Hatier. Due to a recent change of national curriculum in Japan, the textbook will be replaced in the school year 2012. In the analysis, the new textbook obtained as a sample is used. Usually, there is no change after publishing a sample version of textbook.

Four steps of the analysis

The analysis is carried out in the domain of geometry where the proof is introduced in both countries. It consists of four steps. The first step is to identify and clarify what is called 'proof'. In the textbooks, especially in the process of learning geometry, the term 'proof' might not be necessarily used from the beginning, while the other term such as 'justify' or 'explain' could be used. Therefore, I try to identify in this step not only the object called 'proof' but also the objects that are related to the justification, by taking the meaning of proof in a broader sense.

The second step is to identify the main characteristics of the form of proof. What is called 'proof' might not have the same form. A proof given as an exemplary in the textbook will be picked up, and its main characteristics will be discussed. It would allow us to understand what aspect of proof is taken care of as a proof in each country.

The third step is to identify the interrelations of geometrical objects created by means of the proof. One of functions of proof is the systematisation (cf. De Villiers, 1991), that is to say, the creations of interrelations between mathematical objects. I consider that the nature of proof is also characterised by these objects. There would be theorems or properties often used in the proof, and those might be specific to the proof in school mathematics. In this step, I identify the geometrical properties that are proven and the properties or theorems that are employed in a deductive step of proof. In terms of the *praxeology* of ATD, this is to identify *the types of task* appeared in *the genre of task* "prove", and the properties used in *the techniques* to accomplish these types of task (see Chevallard, 1999 for the notion of praxeology). The analysis of this step is conducted mainly on the chapters of textbook where the proof is introduced.

The forth step is to identify the functions of proof in the textbook. Several functions of proof are today well known in the research on proof (cf. De Villiers, 1991). However, in school mathematics, usually, not all of functions, but just some of them could be found. In this paper, I discuss the functions attributed to the proof in particular in the chapters analysed in the prior steps.

RESULTS

In what follows, I report a part of results due to the restriction of pages. The results of the first and second steps of analysis are reported together.

Different proofs and their forms in textbooks

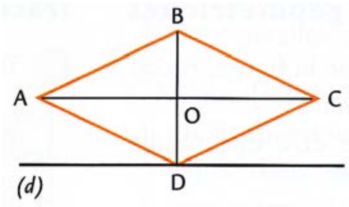
French textbook: In *Triangle* series, one could identify at least two terms for the justification of a statement: "preuve" and "démonstration" (I call "proof" and "mathematical proof" respectively in this paper). These terms can be found from the textbook of the first year of *collège*, *Triangle 6e* (6^{th} grade). However, it is 7^{th} and 8^{th} grades where these terms are explicitly introduced. In Chapter 9 "Initiation to deductive reasoning" of *Triangle 5e* (7^{th} grade), the term "proof" is introduced. This chapter also introduces four rules of 'mathematical debate' with which the truth or false of mathematical statement can be determined. Those are: "(1) A mathematical statement is either true or false. (2) Findings or measures on the drawing do not allow proving that a geometrical statement is true. (3) Some examples that verify a statement is not enough to prove that that statement is true. (4) An example that does not verify a statement is enough to prove that that statement is false. This example is called 'counter-example'" (p. 144). Proof is then a product of this mathematical debate. Fig. 1 shows a proof given as an exemplar in the textbook. As it is for a 'simple' statement which has only a single deductive step, the structure of proof is simple. The conclusion is followed by a colon and the property used to deduce it. The property is stated in the form of "si [if]... alors [then] ...". Most of other proofs that can be found in *Triangle 5e*, in particular in the solutions of exercises at the end of textbook, have the similar form.

Exercise: (d) is the perpendicular bisector of [EF]. (d) cuts (EF) at I. Prove that the point I is the midpoint of [EF].

Solution: The point I is the midpoint of [EF] after the property: "if a line is the perpendicular bisector of a segment, then that line is perpendicular to this segment and passes through its midpoint".

Fig. 1 *Triangle 5e* (2010, p. 145). Only the translation is given due to the restriction of spaces.

In Chapter 8 "Geometry and initiation to the mathematical proof" of *Triangle 4e* (8^{th} grade), the term "mathematical proof" is explicitly introduced. Its definition is: "A mathematical proof in geometry is a succession of the deductive chains which start from the givens and reach at the conclusion" (*Triangle 4e*, p. 147). The deductive chain consists of three elements: given, property and conclusion of chain. Writing of these three elements is emphasised in the textbook, and its instruction is given. The first sentence starts with the phrase "On sait que [We know that]" followed by a given. The second is a conditional statement of the form "Si [If] ... alors [then] ..." as it was in the proof of 7^{th} grade (cf. Fig. 1). And the third starts with a conjunction "Donc [So] …" followed by a conclusion of this deductive chain. The textbook also mentions that the properties could be sometimes left out according to the level of familiarity with them and the teacher's demands (p. 147; p. 149). Fig. 2 shows a mathematical proof given as an exemplar in the textbook, where three elements of a deductive chain can be easily found. One may also notice that the mathematical proof is written as a paragraph, without many mathematical symbols. Most of mathematical proofs given as examples in the textbook or as solutions of exercises at the end of the textbook adopt this form.



Exercise: ABCD is a rhombus with the centre O. Let (d) the line parallel to (AC) which passes through D. Prove [démontrer] that (d) and (BD) are perpendicular.

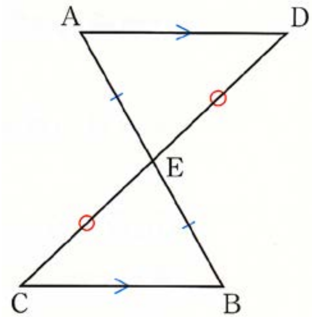
Solution: We know that ABCD is a rhombus.
If a quadrilateral is a rhombus then its diagonals are perpendicular and cut each other at their midpoints.
So (AC) and (BD) are perpendicular.
We know that (AC) and (BD) are perpendicular and that (d) and (AC) are parallel. If two lines are parallel and a third line is perpendicular to the one then it is perpendicular to the other.
So (d) and (BD) are perpendicular.

Fig. 2 *Triangle 4e* (2011, p. 149). Only the translation is given due to the restriction of spaces.

Japanese textbook: In *Tokyo-shoseki* series, the term “proof” can be found in 8th and 9th grades. A part from this term, some exercises require “explanation”. In the 7th grade textbook, the exercise of explanation ask either a description of the procedure of geometrical construction or a kind of justification. In 8th and 9th grades textbooks, some exercises, not many, also require some explanations of justification: “tell/explain the reason ...”, “explain why ...”, etc. The explanation appears before introducing the proof and also after that. However, the method of explanation/justification is implicit in the textbook. Neither instruction nor example is given. One cannot clearly know from the textbook what is really required in explanation. It seems that it is at times the property used in a deductive step, and at other times the given or data used.

On the other hand, the term “proof” is explicitly introduced in Chapter 4 “Parallelism and congruency” of the 8th grade textbook, and the instruction how to prove is given in a sub-section “Method of proving”. The definition of proof given in the textbook is: “showing the reason why a fact is true by means of the properties already known as true is called *proof*” (p. 98). Fig. 3 shows an exemplary proof given in the textbook. The proof is well-organised. Some statements are numbered for the sake of the economy of not restating them later. Mathematical expressions with symbols such as “EA = EB”, “∠AED = ∠BEC” are often used and written separately from Japanese phrases. In this example, the properties used in a deductive step such as “vertical angles are equal” is always written, while the hypotheses or givens are not necessarily stated (e.g., for the statement 3, the hypothesis “AD // CB” is not stated). In the instruction of method of proving, the term “thing that could be grounds” is stressed to be written in a proof (pp. 109-112). It is either a status of statement “hypothesis” or a geometrical property. The form “if … then …” is not used to write a property, while this form is particularly used when claiming a proposition to be proven.

Right diagram is drawn so that the intersection of the segments AB and CD is E, EA = EB, and AD // CB. Let's prove ED = EC.



Proof: In ΔAED and ΔBEC

From hypotheses EA = EB ... (1)

Since vertical angles are equal

∠AED = ∠BEC ... (2)

Since alternate angles of parallel lines are equal

∠EAD = ∠EBC ... (3)

From (1), (2), and (3), since a pair of sides and their extreme angles are equal

ΔAED ≡ ΔBEC

Since corresponding sides of congruent figures are equal

ED = EC

Fig. 3 *Tokyo Shoseki 8th grade* (2012, pp. 109-110). Note: in Japanese secondary school mathematics, the congruencies of segments and of angles are not in the national curriculum. In most of textbooks, the expression "two segments are equal" are used, while it means "two segments have the same length".

Overall, in both countries, proof is an object of teaching, and two kinds of justification appear. However, explanation/justification is not an explicit object of teaching in Japanese textbook, while the proof is. As for the form of proof, the proof of French textbook uses more paragraphs than the proof of Japanese textbook.

Interrelations between geometrical objects

French textbook: The main geometrical properties that are proven in 8th grade where the proof is introduced are parallelism, perpendicularity and midpoint. The former two properties are intensively proven both in the activities for a class and the exercises of the chapter. In addition, at the end of the 8th grade textbook, there are the pages called "Sheet of methods" where the important methods are summarised. Three methods out of five are "Prove that two lines are parallel", "Prove that two lines are perpendicular", and "Prove that a point is the midpoint of a segment". These are main types of task associated with the genre of task "prove" in French textbook. And a list of several properties to be used is given for each method. For example, for the parallelism, six properties are given: "the property of the parallel lines to a same third line", "the property of the perpendicular lines to a same third one", "the property of the opposite sides of a parallelogram, a rectangle, a rhombus, a square", "the property of the line that passes through the midpoint of two sides of a triangle", "the property of the central symmetry", and "the property of alternate-interior or corresponding angles". These are properties that allow accomplishing the three types of task mentioned above.

Japanese textbook: In 8th grade textbook where the proof is introduced, several properties are proven. Among others, the congruency of triangles is quite often proven as a step to prove other properties and plays an important role in the textbook. In Chapter 5 "Triangles and parallelograms" which is a chapter right after introducing the proof, all the theorems introduced in this chapter are proven with a step of proving congruent triangles. Therefore, the congruency of triangles has many interrelations with geometrical theorems or properties, while these interrelations do not exist in French textbook, because it is not an object of teaching.

Besides congruency of triangles, the parallelism which is often proven in French textbook is also proven in Japanese textbook. But it is always by the alternate-interior or corresponding angles. Even in Chapter 7 Section 2 "Parallel lines and ratio" of the 9th grade textbook where Triangle proportionality theorem (Thales' theorem) is an object of teaching, this theorem is merely used for proving a parallelism (only a single exercise out of 22 in this section). That is to say, parallelism is tightly connected to the alternate-interior or corresponding angles in Japanese textbook, while several interrelations are made between parallelism and other properties in French textbook.

Functions of proof

The principle function of proof in French textbook is to justify a mathematical statement as it is written in the textbook: "In order to prove that some geometrical statements are true, one must carry out some mathematical proofs" (*Triangle 4e*, p. 147). The similar remark is given in the descriptions of "mathematical debate". Because the mathematical proof is introduced as an extension of proof or mathematical debate, one can find that the function of communication is one of principal functions of the mathematical proof. This finding conforms to the result of the analysis of proof in French mathematics classroom (Knipping, 2003).

The mathematical statement to be proven in French textbook is either a theorem that can be used in other proofs or a statement only appeared in a particular exercise. In general, it is the proof that allows using the theorem in other proofs. However, this function of proof is less clear in French textbook. A theorem is sometimes admitted first without proof and then its proof is an exercise at the end of chapter (cf. the Midpoint Theorem in *Triangle 4e* Ch. 12, p. 219 and p. 233). Theorems or properties either proven or not proven are summarised in the section of lesson in each chapter. On the other hand, this function can be clearly identified in the Japanese textbook. All the theorems appeared after introducing proof in 8th grade textbook are proven, and their proving is given for the activity in the classroom.

The principle function of proof in Japanese textbook is to justify a mathematical statement, not a particular statement but a general statement. The generality is emphasised. For example, there is a comment with an example of the sum of interior angles in a triangle: "One cannot check out all triangles by means of experiments or measurements, but one can show that the sum of interior angles of any triangle has 180 degrees by means of the proof like the one above" (*Tokyo-shoseki 8th grade*, p. 98). The figures used in proving task also advocate this function. They are not specific ones whose dimensions (length and angle measure) are fixed, but general ones. On the other hand, in the French textbook, the generality is not often emphasised, and even a figure with a fixed dimension is used for proving task. For example, in Chapter 9 "Right triangle and Pythagorean Theorem" of *Triangle 4e*, an exemplar proof is given for the exercise of “Prove [démontrer] that the lines (AI) and (AB) are perpendicular” (p. 164) in which the length of three sides of the triangle ABI are 32, 24, and 40. The solution given to this exercise is called “mathematical proof” in French textbook, while it is rather an explanation in the Japanese textbook (similar exercise and solution can be found in Ch. 6 “Pythagorean Theorem” of *Tokyo-shoseki 9th grade*, p. 155).

CONCLUDING REMARKS

In this paper, I report a part of results obtained in the comparative analysis of French and Japanese textbooks. While some details could not be reported due to the restriction of pages, I expect that the reader can find some differences on the nature of proof in two countries. The proof in the textbook is a proof *to be taught*. Its differences would imply different consequences in the teaching and learning of proof in the classroom of different countries. The nature of *taught* proof is a further question, in addition to the question on the system of conditions and constraints that forms the nature of proof.

Acknowledgements

This work is partially supported by KAKENHI (23730826 and 22330245).

References

Artaud, M. (1998). Introduction à l’approche écologique du didactique – L’écologie des organisations mathématiques et didactiques, In M. Bailleul, et al. (Eds.), *Actes de la IXème école d’été de didactique des mathématiques* (pp. 101-139). Caen: ARDM & IUFM.

Balacheff, N. (2008).The role of the researcher’s epistemology in mathematics education: an essay on the case of proof. *ZDM*, 40(3), 501-512.

Cabassut, R. (2005). *Démonstration, raisonnement et validation dans l’enseignement secondaire des mathématiques en France et en Allemagne*. Thèse, Université Paris 7.

Chapiron, G. et al. (2009, 2010, 2011, 2008). *Mathématiques 6e, 5e, 4e, 3e, Collection Triangle*. Paris: Hatier.

Chevallard, Y. (1991). *La transposition didactique* (2nd ed.). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1994) Les processus de transposition didactique et leur théorisation, In G. Arsac et al. (Eds.) *La transposition didactique à l’épreuve* (pp. 135-180). Grenoble: La Pensée Sauvage.

Chevallard, Y. (1999). L’analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, 19 (2), 221-266.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17-24.

Fujii, T. et al. (2012). *Atarashii Suugaku 1, 2, 3* (sample version). Tokyo: Tokyo-shoseki.

Knipping, C. (2003) Processus de preuve dans la pratique de l'enseignement - analyses comparatives des classes allemandes et françaises en 4ème. *Bulletin de l'APMEP*, 449, 784-794.

Reid, D. A. & Knipping, C. (2010). *Proof in mathematics education*. Rotterdam: Sense Publishers.