REFLECTIVE SYMMETRY IN CONSTRUCTION AND PROVING

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The aim of this research is to advance understanding of how mathematical knowledge functions in the proving in geometry. We focus on the rules whose mobilization is due rather to the mathematical knowledge at stake than to the proof. We observed students who are asked to solve construction problem and proving problem. The problems require to mobilize the "same" rule from the theoretical point of view. The first result is that even if the students can construct correctly a symmetric and are conscious of the necessity of perpendicular and equal distance, it doesn't mean that they can use the rule appropriate in proving.

INTRODUCTION

What relationship does it exists between the knowledge about a specific mathematical notion and the nature of proof? There would be a strong relationship, as many researches report the role of proof for mathematical understanding (cf. Hanna, 2000). Our research interest is, in a sense, inversive: try to make clear not the role of proof for knowledge but the role of knowledge for proving. In this paper we focus on the state of knowledge in geometrical construction and proving and propose an explicitation for their relationship or differences.

We have selected a precise domain in geometry, the reflective symmetry[1]. The reason for this choice is first that many researches have been done on symmetry, especially for the construction and recognition problems in the paper-pencil environment as well as computer-based (Küchemann, 1981; Grenier, 1989; Bell, 1993; Hoyles & Healy, 1997; etc.). It will facilitate the analysis of students' knowing of reflection. The second reason is that the reflection is rarely involved in proving problems in the class. This means that the students of our observation should rely on and mobilize the knowledge acquired in non-proving context.

THEORETICAL FRAMEWORK AND RESEARCH METHOD

The structure of proof or reasoning can be expressed by the triad: given statement, rule of inference, and conclusion (Duval, 1991; Figure 1). As the rule of inference connects two statements, it can be expressed in the form of an implication "If A then B". So, a proof will be the combination of some triads like in a graph. Of course, some of these triads – the obvious ones in the eyes of the readers – might not be written in the proof. A model developed by Toulmin (1958) to analyze argumentation also has a similar structure. Referring to this structure, we consider that the knowledge involved in reasoning comes within the rule of inference, because statements cannot be

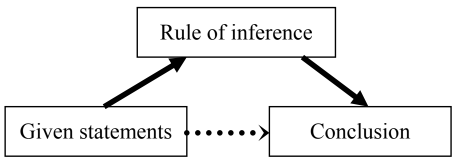


Figure 1 Structure of reasoning by Duval

connected without it.

We also consider the hypothesis that people do not have rules as a long and huge list, in other words, rules are also based on something which support them. Toulmin (1958) call it "backing". In the words of cK¢ model developed by Balacheff and his research group (Balacheff & Gaudin, 2003; Balacheff, 2000; etc.), this is a conception or instantiated knowledge. We use, as a tool of analysis, the cK¢ model which gives us certain points of view for analyzing not only the proof but also the students' products and behavior from an epistemic point of view. In this model a conception is characterized by the quadruplet (P, R, L, Σ): the set of problems on which the considered knowledge is efficient, the set of operators involved in the problem solving activity, the system of representation used (i.e. semiotic system), and a control structure. We pick up two aspects from this description of a conception: operators which can correspond to the rule of inference in the proof structure and control structure which is behind the decision to use operators and validates them. We call in this paper "rule" what can be expressed as "if A then B" which would be mobilized as a rule of inference or operator.

The questions concerning the rules have been studied, with the term "conditionality of the statements", by the Italian research group (Boero et al., 1996; 1999; etc.). They analyzed the process of its generation, and its link to proving process. Our study would be situated to a more basic problematic with the notion of symmetry, that is, diagnosis of students' state of knowledge from the rule point of view. The questions posed are:

- What rules may exist for the symmetry?
- Which rules may be mobilized in the construction and proof problems? Do they have the same nature?

To reply to these questions, we take the following process. First, we give a theoretical analysis of rules involved in the use of symmetry. Second, we analyze the construction process of symmetric to identify rules which may be mobilized. Based on these results, we propose a proving problem which requires the same rule as construction, and organize an observation experiment for the 9th grade students. We present the data collected in the observation and a case study with specific students.

ANALYSIS OF RULES FOR THE REFLECTIVE SYMMETRY

We analyze the possible rules generated for the symmetry. From this analysis we construct a framework for analyzing the data obtained in the observation. Three types of rules concerning symmetry can be formalized as a form of implication with respect to the relationship between symmetry and geometric properties or objects.

Type 1. (symmetry $\Rightarrow$ property) rule connecting symmetry to an geometric property:

R_1: If P_2Q_2 = Sym (P_1Q_2, d) then $P_1Q_1 = P_2Q_2$[2]

R_2: If P_2 = Sym (P_1, d), then $P_1P_2 \perp d$

R_3: If P_2 = Sym (P_1, d), then $P_1M = MP_2$ ($M \in d$);

Type 2. (properties $\Rightarrow$ symmetry) rule connecting some geometrical properties to symmetry. This is a rule which characterizes the symmetry:

R_4: If $P_1P_2 \perp d$ and $P_1M = MP_2$, then $P_2 = \mathrm{Sym}(P_1, d)$ ($M = d \cap P_1P_2$)

R_5: If $P_1M = MP_2$ and $P_1N = NP_2$, then $P_2 = \mathrm{Sym}(P_1, d)$ ($M \neq N \in d$);

Type 3. (symmetry $\Rightarrow$ symmetry) rule expressing symmetrical relationship:

R_6: If $P_2 = \mathrm{Sym}(P_1, d)$ and $P_4 = \mathrm{Sym}(P_3, d)$, then $P_2P_4 = \mathrm{Sym}(P_1P_3, d)$.

R_7: If $P_2P_4 = \mathrm{Sym}(P_1P_3, d)$, then $P_2 = \mathrm{Sym}(P_1, d)$ and $P_4 = \mathrm{Sym}(P_3, d)$.

The above list of rules is not exhaustive at all. These are not always of the form "if A then B" when being mobilized. But we try to identify them in the resolution process and students' statement collected in the protocol. For example, when one says a property of the symmetry such as "two symmetric segments have the same length", we understand that the rule "R_1: If $P_2Q_2 = \mathrm{Sym}(P_1Q_2, d)$ then $P_1Q_1 = P_2Q_2$" is used.

CONSTRUCTION FROM RULE POINT OF VIEW

We analyze here which type of rules are mobilized in the correct construction process of symmetry. This analysis makes clear the relationship between construction and proof from the rule point of view. In fact, we will be able to make a proving problem with a rule needed in the construction.

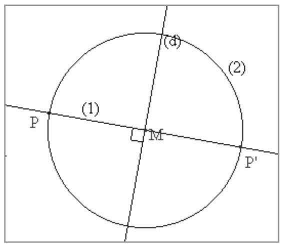


Figure 2

The usual construction for a symmetric point would have the procedures like Figures 2 and 3. These two procedures can usually be found in the school textbook. We analyse these cases. The first one is to draw first (1) the perpendicular line PM to the axis *d* through a given point P, and (2) take the distance PM to the "other side". For the perpendicular line, the triangular or compass and for the equal distance, the compass or ruler with graduation can be used as instruments. The operators mobilized in this construction are the actual construction procedures, (1) and (2), which give a point P' on the paper. They would be based on "R_4: If $P_1P_2 \perp d$ and $P_1M = MP_2$, then $P_2 = \mathrm{Sym}(P_1, d)$". In fact, mathematically, the perpendicular line PM is drawn and the point P' which is at the equal distance as PM is taken, since these two geometrical properties give a symmetric point.

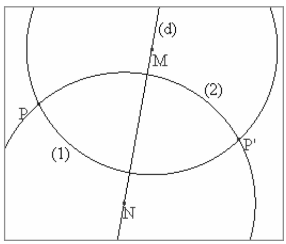


Figure 3

The second procedure (Figure 3) is to draw two circles whose centers are on the axis, and the intersection P' is symmetric of P. It is based on "R_5: If $P_1M = MP_2$ and $P_1N = NP_2$, then $P_2 = \mathrm{Sym}(P_1, d)$". In fact, two circles are drawn to get a set of equidistant points from two points on the axis, and two conditions are enough to determine one point on the plane, so two equal distances gives a symmetric point.

From this analysis, we can find that the type of rules needed for the construction is type 2 (properties $\Rightarrow$ symmetry) which characterizes a symmetric point. We also find that in the construction, the rule is not operational but predicative. In fact, it's not rule but actual action that draws a symmetric on the paper. In the words of cK¢ model, they are used not as an operator but as a control. So, two aspects are possible for a same symmetric "element".

PROBLEMS IN THE OBSERVATION

We organized an observation to see how students solve construction and proving problems which require a same rule, type 2, R_4. We present here two of four problems given in the observation.

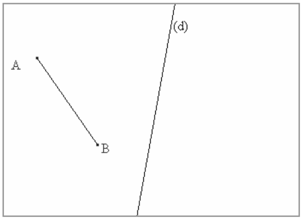
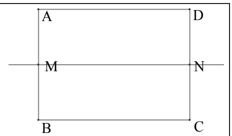


Figure 4

Problem 1 is usual one. We ask to draw the symmetric segment of AB with any usual instruments (ruler with graduation or not, compass, triangular, protractor, etc.). Any instruments can be used, because we didn't want to give restriction for constructing other geometric objects than symmetric. What we want to know is the geometric properties mobilized for the construction of symmetry, but not the techniques on the instrument, nor the mobilized properties for the construction of perpendicular line or equal distance. For the given figure, we took into account some didactical variables identified by Grenier (1987), such as the slope of axis, the direction of segment with respect to the axis, etc., in order to appear the incorrect procedure. As demonstrated by the analysis presented in the previous section, to construct the symmetric point of A, one of type 2 rules should be used. And it's same for B. After drawing the symmetric points A' and B', it needs another rule which allows to draw A'B' as a symmetric segment of AB. It would be a type 3 rule "R_6: If P_2 = Sym (P_1, d) and P_4 = Sym (P_3, d), then P_2P_4 = Sym (P_1P_3, d)".

Problem 2 is to recognize two symmetrical segments and to prove it. Since this is a proving problem for the characterization of symmetry, the type of rules which should be mobilized is type 2, as in the problem 1. In fact, to prove that the segments AD and BC are symmetric, it is needed at first that extremities of segments are symmetric, such as "AM = MB and AB $\perp$ MN, so A and B are symmetric". After the proof for extremities, the proof of the symmetry of segments requires the R_6 rule.

Problem 2: *The quadrilateral ABCD is a rectangle. Let M, N, be the midpoints of opposite sides AB et DC. Are the segments AD et BC symmetric with respect to the line MN? Reply with "yes", "no", or "not always", and prove your answer.*	

We have observed 25 students of 9th grade (aged almost 14 years) in France: we had 11 pairs, and only one group of three who are asked to work together and give one answer. This observation method aims at expliciting students' conception as far as possible. The observers of each pair noted students' behaviors.

In France, proof learning begins progressively from explanation or justification at the 6th grade and is taught mainly in relation to deductive reasoning at the 8th grade in geometry. The notion of symmetry is introduced at 6th grade with construction. After 6th grade, it is often used as a tool to analyze other geometric objects. The reflection is not often used in the proving context.

THE DATA FROM THE OBSERVATION

We give at first the data of students' answers. For the problem 1, most pairs give a correct construction with a triangular, compass, ruler with graduation or not. In the table below, the pairs using triangular drew perpendicular line to the axis *d*. The compass is used for the equal distance, while the pair 10 uses ruler with graduation for it. Only the pair 2 gives an incorrect construction. The line drawn is not perceptively perpendicular to the axis *d* and the instrument used does not have perpendicular. For the problem 2, all pairs give a correct answer "Yes", two segments are symmetry. But the pair who gives a correct proof is only the pair 9.

As we analyzed it in the previous section, the rules which should be used are same. It seems that if the students use the triangular and the compass for the construction of symmetric point, they use the type 2 rule R_4. However, our data shows that most students who gave a correct construction do not give a correct proof, even if the same rule is needed. The questions posed now are "What is the difference?", "Why can't they give a correct proof?", and especially concerning the type 2 rule "Do they really have the type 2 rules?". We try to answer these questions.

Pair	Problem 1: instruments	Problem 2
1	Triangular, compass	Yes
2	ruler, compass	Yes
3	Triangular, compass	Yes
4	Triangular, compass	Yes
5	ruler, compass	Yes
6	Triangular, compass	Yes
7	Triangular, compass	Yes
8	Triangular, compass	Yes
9	Triangular, compass	Yes
10	Triangular, ruler with graduation	Yes
11	ruler, compass	Yes
12	Triangular, compass	Yes

A CASE STUDY

We are going to analyze the products and protocols of the first pair, Delphine & Baptiste (1), from the rule point of view. We identify the rules mobilized in the protocol for the construction and proving problems, and analyze the related controls.

Construction for the problem 1 by Delphine & Baptiste (1)

Their construction would be accepted by most teachers at the secondary school (Figure 5). They draw the perpendicular line with triangular and take equal distance with compass for A and B separately, and connect two points A' and B'. Then, they mark some codes for equal distance and right angle which allow us to diagnose that they are conscious of the perpendicular attached to triangular and the equal distance to compass, and their necessities for the symmetry. It's also explicit in the protocol.

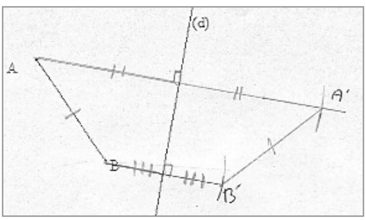


Figure 5 Construction by pair 1

11. Baptiste: this one, you have to make it symmetrical.

12. Delphine: yah. You draw a perpendicular segment to the line d, passing through B. Then with your compass,

(…)

24. D: then, you make the same thing for … there, it's not perpend … you are doing ... what are you doing!?
25. B: what? I'm drawing
26. D: no! It should be a right angle, there.
27. B: it's perpendicular.
28. D: it should be perpendicular.
29. B: hummmm,
30. D: it's not perpendicular. Do you know why it should be perpendicular?

(…)

36. D: you see, in fact, it should be the same segment, when one sees in front of the mirror.
37. B: hum.
38. D: It makes us an inverse. In fact, it's inverse. So, it should be the same slope with respect to the line d.
39. B: OK, I see.
40. D: to be the same slope, well, it should be perpendi …

In the protocol, it seems that Baptiste is not conscious of the necessity of perpendicular. But Delphine clearly states it and "right angle" [26; 28; 30]. After [30], Delphine tries to explain its necessity, but her explanation is not clear [36; 38; 40]. It relies on some examples given by the perceptive definition "mirror effect" [36]. In the words of cK¢ model, the necessity of perpendicular is validated by a perceptive control "mirror effect". We also consider that the construction with perpendicular is one justification for the necessity of perpendicular, such as "symmetric segments which satisfy the perceptive control could be constructed with perpendicular, so it is necessary for the symmetry".

Well, which rule do they mobilize? As the perpendicular and equal distance are used, can we interpret them as the type 2 rule "R_4: If $P_1P_2 \perp$ d and $P_1M = MP_2$, then $P_2 =$ Sym (P_1, d)" is mobilized? From the construction, it seems that they do. But, we will find from the problem 2 that the answer is rather negative than positive.

Proof for the problem 2 by Delphine & Baptiste (1)

For the problem 2, they construct a proof in which two following rules can be identified:

R_{DB1}: "if $P_1P_2 \perp P_2P_3$, $P_4P_3 \perp P_2P_3$ and $P_1P_2 = P_4P_3$, then P_1P_4 // P_2P_3"

R_{DB2}: "if Q_1Q_2 // d // Q_3Q_4 and $Q_1Q_2 = Q_3Q_4$, then Q_1Q_2 = Sym (Q_3Q_4, d)"

The first rule R_{DB1} appears twice in the first eight lines. It is not correct rigorously, because P_1P_4 and P_2P_3 are not parallel if P_1 and P_4 are at the different side each other with respect to the line P_2P_3. The second rule R_{DB2} is identified in the last sentence (last four lines). This is not also correct.

Yes
(AM) ⊥ (MN)
(DN) ⊥ (MN)
[AM] = [DN]
thus (AD) // (MN)
(BM) ⊥ (MN)
(NC) ⊥ (MN)
[BM] = [NC]
thus (BC) // (MN)
As the segments [BC] and [AD] are parallel to (MN) and they are the same length, then they are symmetric with respect to the line (MN).

As they use the perpendicular "AM ⊥ MN" as a hypothesis (while this property is not stated in the problem statement) and the equal distance "AM = MB" in the proof, we can find that they have enough properties for characterizing symmetry. But they don't mobilize the rule "R_4: If $P_1P_2 \perp d$ and $P_1M = MP_2$, then $P_2 = Sym(P_1, d)$".

62. D: For me, I think "Yes". I explain you why. Because you see there, there is a right angle with respect to this line.
63. B: they are equal.
64. D: yes, and a right angle, here, they are parallel.

(…)

68. D: yeah, in fact, you see, MA is equal to MB. You see?
69. B: yes, yes, yes, I understand it.
70. D: so, as they are both the perpendicular segments

(…)

76. D: and the line AB is perpendicular to the line MN.
77. B: hum.
78. D: and the line BC is perpendicular to the line MN. They are both perpendicular to the same line, and MA is equal to MB. So AD is parallel to MN.
79. B: hum. Yes, yes, I understand.
80. D: On the contrary, how shall we write? I don't know.

In the protocol, it's clear that Delphine notices perpendicular or right angle [62; 64; 70; 76] and equal distance [68; 78] and the right angle is identified as an important criterion for symmetry [62]. But she doesn't notice the rule R_4. At the end of proving process, the rule R_{DB2} is mobilized without discussion [117-118]. This time, parallel and equal are stated together by Baptiste.

117. B: parallel and equal. It's symmetric.
118. D: that's right. I leave you to write down a little.

DISCUSSION

In the previous sections, we presented the data obtained in the observation and analysis from rule point of view. The first result is that most students who gave a correct construction cannot give a correct proof, even if the same rule is required. Moreover, although they are conscious of the properties necessary for proving, it's not same as using a correct rule. In the case of Delphine & Baptiste (1), even if they can construct correctly and are conscious of the necessity of perpendicular for symmetry, it doesn't mean that the appropriate type 2 rule (R_4 in the problem 2) can be use. Thus, the construction with geometric instruments is not enough to acquire and mobilize rules characterizing symmetry.

Why don't Delphine & Baptiste (1) mobilize R_4 in proving? Don't they have it? We consider that they don't have the rule R_4. One of the reasons is that the perpendicular and equal distance are mobilized separately in both problems, while they should be together to be a rule and utilizable in the proof. In fact, in the construction (problem 1), Delphine doesn't talk about the necessity of perpendicular and equal distance together and never gives statement characterizing symmetry. For example, her statement "it should be perpendicular" often appeared [26; 28; 30] is not based on the type 2 rule but on the type 1 rule "R_2: If $P_2 = Sym(P_1, d)$, then $P_1P_2 \perp d$". She also states "it should be the same segment" [36], "it should be the same slope" [38]. These statements have the same nature as for the perpendicular [26; 28; 30], because these are properties of symmetry and imply the type 1 rules, not the type 2. In the proving (problem 2), as we have already mentioned, the right angle is an important criterion for symmetry for Delphine. But it's not also together with equal distance. In the proving process, while they found many geometrical properties – perpendicular or right angle [62], same length [63], parallel [64], same distance [68; 78] –, "parallel" and "same distance" which allow us to diagnosis the rule R_{DB2} are qualified for proving at the end. Therefore, we understood that the students do not have the appropriate type 2 rule R_4.

We should also consider the role of rule in two problems. As we have found in the theoretical analysis, the rule for the construction is mobilized in a predicative form, while it should be operational in proving, even though the same aspect of symmetry is dealt. So, the construction without explicit rule, but with some separated geometric properties would be accepted in the school context, while proving requires the formalized and operational rule.

NOTE

1. In this paper, when we say "symmetry", it means reflective symmetry.
2. "P_2Q_2 = Sym (P_1Q_1, d)" means, in this paper, that two segments P_1Q_1 and P_2Q_2 are symmetric with respect to a line d. The order of P_1Q_1 and P_2Q_2 are indifferent, while it would be different from the cognitive point of view. And we use this notation also for the point.

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