RELATION BETWEEN PROOF AND CONCEPTION: THE CASE OF PROOF FOR THE SUM OF TWO EVEN NUMBERS

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The aim of this research is to advance understanding of how mathematical knowledge functions in the proof construction, especially in its written outcome with a problem in algebra. The theoretical analysis allows us to explain some reasons of students' proofs and their tendency obtained by a case study. The first result is that the difficulty of constructions of mathematical proof is due not only to the algebraic competence or proof conception, but also to the mathematical knowledge.

INTRODUCTION

In this paper we report some finding from an analysis of students' proofs in algebra taking into account mathematical knowledge at stake. The aim of this research is to advance understanding of how knowledge functions in the proof construction, especially in its written outcome. This is to know which relation may exist between the mathematical knowledge involved and the nature of proof.

Our research question comes from the recent study about proof conceptions by Healy & Hoyles (1998, 2000). These authors found, from their grand survey in the Great Britain with a statistic method, students' proof conception in algebra that proofs constructed by students follow an empirical approach or a narrative style rather than a formal one although most students are aware of their limitations (2000, p.396). The authors discussed that this problem was due to the lack of algebraic understanding for the proof (2000, pp.425-426). We share the idea that this is one of the possible reasons, but we consider that this is not the unique one. We develop this point in this paper by focusing on the mathematical knowledge. We intend to evidence the role played by students' conception of the mathematical notion (not proof conception) in their proof construction and learning as well.

We take for our analysis the proof problem of sum of two even numbers already studied by Healy & Hoyles. We partially replicate this study (on a smaller scale) and show how one can interpret the data gathered.

THEORETICAL FRAMEWORK

Proof

To characterise or classify students' approaches to proving, some research results have already been presented: proof types of Balacheff (1987), proof schemes of Harel & Sowder (1998), characterisation by the structure of reasoning (Duval, 1991), "proofs that prove" and "proofs that explain" of Hanna (1989), etc. In this paper, we adopt as a theoretical framework, the proof types of Balacheff (1987) from "pragmatic proof" to "intellectual proof", more precisely four types of proof – "naive empiricism", "crucial experiment", "generic example", and "thought experiment" – and "mathematical proof" (also called "formal proof"). We expect this choice to be relevant in order to characterise the relationship between the nature of proof and that of knowledge. In fact, to characterise these types of proof Balacheff takes into account not only the nature of the underlying rationality and the language level, but also the nature and the status of knowledge (1987, p.163), which is the most important point of view in our analysis.

Conception

To identify and differentiate students' knowledge of even number involved in their proofs, we adopt the notion of conception. Conception is a didactical tool for modelling the students' knowledge in problem solving situations. It reveals the plurality of the possible points of view on a same mathematical object, it differentiates the representations and related methods, and emphasises their more or less good adaptation at the resolution of such and such class of problems (Artigue, 1990, p.265). In our analysis, we pick up aspects as the operators which are used to solve a problem, and the language which is also important from the perspective of proof characterisation[1].

The conceptions of even numbers we can identify, considering operators appearing in schoolbooks or in students' proofs are following. From C_2, we can get out three operators by the concept of divisibility or multiplicity. C_3 is an algebraic one.

C_1: Even numbers end by 0, 2, 4, 6, or 8.
C_2: Even numbers can be divided by 2, or the result of the multiplication of a whole number by 2[2].
 C_{21} : Even numbers can be divided in two identical parts.
 C_{22} : Even numbers can be decomposed 2 by 2.
 C_{23} : Even numbers have 2 as a factor.
C_3 : Even number can be expressed by $a = 2p$ (p: whole number).

ANALYSIS OF THE POSSIBLE PROOFS FOR EACH CONCEPTION

By this analysis we intend to construct a framework for discussing the data obtained during the observation. We try to construct several different proofs from "pragmatic" to "intellectual" involving the conceptions we have presented. This analysis allows us to locate or to characterise students' proofs, and also to identify the relation between the conceptions of even number in the proof. Concerning the representations used to express proofs, we pay attention to the operational one[3], that is, the representations on which computations or transformations can be carried out. In fact, the natural language is often used in the formulations of proof, but we should often take care to distinguish it from the operational one.

List of proofs

C_1: Even numbers end by 0, 2, 4, 6, or 8.
Naive empiricism (numerical): 2 + 4 = 6, 4 + 8 = 12, 6 + 8 = 14, 12 + 24 = 36.

Crucial experiment (numerical): I take arbitrary two even numbers. 188+ 76 = 264.
Generic example (numerical): With two even numbers: 18 and 24. 18 + 24 = (10 + 8) + (20 + 4) = (10 + 20) + (8 + 4) = (10 + 20 + 10) + 2. We can do this for any two even numbers.
Thought experiment (natural language): Even numbers end by 0, 2, 4, 6, or 8. The last digit of number of sum of two numbers is calculated by the sum of their last digit of numbers. So the sum of two even numbers ends also by 0, 2, 4, 6, or 8.
Mathematical proof (numerical): The proof is same as that of thought experiment but with the table 1.

	0	2	4	6	8
0	0	2	4	6	8
2		4	6	8	0
4			8	0	2
6				2	4
8					6

Table 1: exhaustion

C_{21}: Even numbers can be divided in two identical parts
Naive empiricism (numerical): 4 + 8 = 12 = 6 + 6, 6 + 8 = 14 = 7 + 7, etc.
Crucial experiment (numerical): I take arbitrary two even numbers. 188 + 76 = 264 = 132 + 132.
Generic example (numerical): With two even numbers: 12 and 18. 12 + 18 = (6 + 6) + (9 + 9) = (6 + 9) + (6 + 9) = 15 + 15. We can do this for any two even numbers.
Generic example (graphical): **:::** + **:::::** = **::::::::** (separate horizontally). We can do this for any two even numbers.
Thought experiment (natural language): Even numbers can be divided in two identical parts. So, if you add each divided part of two even numbers, the sum can be also expressed by two identical parts.
Mathematical proof (algebraic): $\forall$a, b: even number, $\exists$ p, q $\in$ **Z** s.t. a = p + p, b = q + q. a + b = (p + p) + (q + q) = (p + q) + (p + q).

C_{22}: Even numbers can be divided 2 by 2
Naive empiricism (numerical): 4 + 8 = 12 = 2 + 2 + 2 + 2 + 2 + 2, 6 + 8 = 14 = 2 + 2 + 2 + 2 + 2 + 2 + 2, etc.
Crucial experiment (numerical): I take arbitrary two even numbers. 188 + 76 = 264 = 2 + 2 + ….
Generic example (numerical): With two even numbers: 4 and 8. 4 + 8 = (2 + 2) + (2 + 2 + 2 + 2) = 2 + 2 + 2 + 2 + 2 + 2. We can do this for any two even numbers.
Generic example (graphical): **:::** + **:::::** = **::::::::** (separate vertically). We can do this for any two even numbers.
Thought experiment (natural language): Even numbers can be divided 2 by 2. So, if you add two numbers divided 2 by 2, the sum can be also expressed 2 by 2.
Mathematical proof (algebraic): $\forall$a, b: even number, $\exists$ p, q $\in$ **Z** s.t. a = 2 + 2 + … + 2 (p terms), b = 2 + 2 + … + 2 (q terms). a + b = (2 + 2 + … + 2) + (2 + 2 + … + 2) = 2 + 2 + … + 2 (p + q terms).

C_{23}: Even numbers have 2 as a factor
Naive empiricism (numerical): 4 + 8 = 12 = 2 $\times$ 6, 6 + 8 = 14 = 2 $\times$ 7, etc.
Crucial experiment (numerical): I take arbitrary two even numbers. 188 + 76 = 264 = 2 $\times$ 132.

Generic example (numerical): With two even numbers: 12 and 32. $12 + 32 = 2 \times 6 + 2 \times 16 = 2 \times (6 + 16) = 2 \times 22$. We can do this for any two even numbers.
Thought experiment (natural language): Even numbers have 2 as a factor. So, if you add two numbers having 2 as a factor, the sum have also 2 as a factor.
Mathematical proof (algebraic): $\forall$ a, b: even number, $\exists$ p, q $\in$ **Z** s.t. $a = 2 \times p$, $b = 2 \times q$. $a + b = 2 \times p + 2 \times q = 2 \times (p + q)$.

C_3 : Even number can be expressed by $a = 2p$ (p : whole number).
Mathematical proof (algebraic): $\forall$ a, b: even number, $\exists$ p, q $\in$ **Z** s.t. $a = 2p$, $b = 2q$. $a + b = 2p + 2q = 2 (p + q)$.

What we can expect from this analysis

Distance between the conceptions: C_1 tests all the cases, C_2 and C_3 show structure

We can point out some differences between C_1, C_2, and C_3 by analysing each type of proof from a mathematical point of view. C_1 pays attention to numbers or the representation expressed by the decimal system[4] and observes all the sums of last digit of numbers between 0, 2, 4, 6, and 8 (exhaustive). In this case, the structure of even numbers or their sum is not evidenced. On the contrary, C_2 and C_3 do not focus on specific numbers, but the structure of even numbers, of their transformations, and of their sums, and the proofs are constructed by showing them.

Whereas we gave a proof based on a generic example with C_1, this example is not for sums of numbers chosen among {0, 2, 4, 6, 8}. In fact, C_1-proofs except naive empiricism and crucial experiment can be separated in two phases. The generic example showing the structures intends to establish the first phase "the last digit of sum of two numbers can be calculated by the sum of each last digit of numbers". The second phase consists of showing that the sums of two digits between {0, 2, 4, 6, 8} end by one of the same digits. Thus C_1 cannot give generic example for a whole proof, although C_2 can. This division in two phases also shows a gap between pragmatic proofs having only one phase and more intellectual proofs. In fact, not limiting at this example of two even numbers, it's not easy for the students to shift from a mere judgement to eliciting the structure of a mathematical notion.

Besides, we can also find that C_2 and C_3 have a very close nature. In fact, C_{21} divide into two identical parts ($p + p$), C_{22} divides 2 by 2 (2 + 2 + … + 2 (p terms)), and C_{23} has 2 as a factor ($2p$). And these "2" appearing in C_2 can be seen as "2" of "$2p$" in C_3. Thus we consider that there is not a big obstacle in passing from C_2 to C_3. On the contrary, C_1 where only the last digits are important is very different from C_2 and C_3. From these points of view, we can find that the distance between C_2 and C_3 is smaller than between C_1 and C_2.

Three elements relying on each other: conception, representation, and proof type

Among the different proof types we have presented, the construction of some proofs is a little probable. For example, C_2-naive empiricism and C_2-crucial experiment. Almost everybody would use C_1 when they are asked to judge whether the given number is even or not (ex. Is the number 243992 even?). It's enough to verify the last digit of number and very easy and simple. On the other hand, judging a large number by C_2 requires a great and hard work. As the naive empiricism and the crucial experiment require this simple judgement, if the students have a proof conception of these types, it's obvious that they take C_1. Thus, we could state that more relevant or effective conception exists according to the choice of a proof type.

On the contrary, we can also state that the choice of a proof type depends on the "available conception"[5]. As C_2 already shows the structure of even numbers, this conception facilitates the construction of a generic example. And it's very obvious that nothing but a mathematical proof can be constructed with C_3.

As concerns the language, the representation used in a C_1-proof is likely to be numerical since C_1 focus is always on the last digit expressed by the decimal system. Thus, even if students have algebraic representation as a modelling tool, it will be difficult for them to produce an algebraic mathematical proof if they have not available a relevant conception (in our case, C_2 or C_3). In other words, if the available conception for students is only C_1, it is unlikely that they will construct an algebraic proof. On the other hand, the possible representations with C_2 could be numerical, graphical, algebraic, and the natural language, because C_2 doesn't always focus on the digits like C_1. And, only the algebraic representation is used for C_3. This key role of language is not new. However what is remarkable is that it's not always the algebraic representation which is necessary to construct a mathematical proof. The most relevant one for a mathematical proof depends on the mobilised conception. Therefore, if the intention of teacher is to make students to construct an algebraic proof, he has to design a situation which "disqualifies" C_1 but favour C_3.

The graphical representation that appears to be a rather easy way to present the structure of a number and a given operation, is only plausible under C_{21} and C_{22}. We suggest that this representation may be used with the intention to make more "visible" the structure (like "proof that explain" of Hanna (1989)). On the contrary, one can remark that the representation impacts the choice of a proof type. Let us take C_2 as an example. If one use only the numerical one, the possible types of proof are naive empiricism, crucial experiment, and generic example. With natural language, only thought experiment, with graphical one, generic example, and with algebraic one, mathematical proof.

THE RESULTS OF A CASE STUDY

Observation

To get data about proofs constructed by students, we have proposed a questionnaire to 37 students of 9^{th} grade (aged almost 14 years) from Grenoble area in France. The students were just asked to make a proof for the sum of two even numbers: "Is the following statement is true or false? Prove your answer. «When you add any two even numbers, your answer is always even»". We didn't present examples of proofs to students like in the study made by Healy & Hoyles (2000), because the aim of our observation is to get some data in France and to characterise their tendency and their possible reasons. In France, the proof learning begins progressively from explanation or justification at the 6th grade and is taught mainly in relation to deductive reasoning at the 8th grade in geometry.

Results

We have classified students' proofs depending on the conceptions and the types of proofs (table 2). The criteria of this classification are following:

	C_1	C_{22}	C_{23}	Other
Naive empiricism	12			
Crucial experiment	8			
Generic example	1	4		
Thought experiment			1	
Mathematical proof	3		2	
Other			2	4

Table 2: Frequencies for conceptions and proofs

Naive empiricism: proofs based on few cases, only some sums are shown.
Crucial experiment: these proofs consist of a statement that describes the use of large numbers like "with two large numbers", "take arbitrary two numbers", or large numbers explicitly used comparing to others, even if it is not stated.
Generic example: these proofs consist of just one case that is specifically analysed and this analysis attempts to show the structure of even numbers and their treatment.
Thought experiment: such proofs attempt to show the structure of even numbers and their transformation or computation with natural language.

These proofs can not always reach a complete achievement. Some proofs in which we were not able to identify a related conception, are classified to the column "other". And the line "other" of proof type is for the case where the statement "the sum is even" is taken as a hypothesis (circular argument), or where students evidence their conceptions but don't show any idea of a proof. We couldn't find C_{21}-proof or C_3-proof. As regards the operational representation, only one student used the natural language (C_{23}-thought experiment). Very few students used the algebraic representation (2 C_{23}-mathematical proofs and 1 C_{23}-other). No student used the graphical one. And all the rest took the numerical one (hence 33/37). Whereas the observation method was not same, we couldn't remark, as Healy & Hoyles (1998, 2000), that students used narrative argument for their own approach.

Much more than half of the students (24/37) produced a C_1-proof, most of them (20/24) based on a naive empiricism or a crucial experiment. And proofs of these types were only based on C_1. We consider two mutual reasons of these frequencies following our theoretical analysis. First, proof conceptions of students remain in the types of naive empiricism or crucial experiment, like Healy & Hoyles stated, so they take C_1 that is useful for these types. The proof types precede the conceptions. Second, the conception C_2 is not available to students, so they take C_1 which lead to these types of proof. The conceptions precede the proof types. In this second reason, it is not same what they "know" about a tool as they can use it as own tools. For example, Maud's proof (figure 1) shows two conception: "the even numbers go two by two" (C_{22}) and "they always end by same digit" (C_1). But the conception used in the given example is only the later. With these two reasons, we cannot explain why they cannot give intellectual proof. We consider that this is due to the different characters between the proof types. As we have mentioned in our analysis of the possible proofs, there is a gap in the proof structure between the first two types and the others: the former has one phase and the latter has two phases. Among these reasons, none relates to the lack of algebraic knowledge, that is, even if students had algebra available, they could not construct an algebraic mathematical proof with C_1.

Only one seems a proof based on a generic example (figure 1). We have classified this proof as a generic example, whereas the treatment of last digits was not shown. In this proof the second phase (exhaustive sum) was implicit. Maud would think the second phase was too evident. On the contrary, all C_1-mathematical proofs were proof by exhaustion and the first phase was always implicit, that is, students would think the first phase was too evident. This point gives evidence of difference what must be proved between students.

> The statement is true, because the even numbers go two by two; they always end by the same digit: 2, or 4, or 6, or, 8, or 0.
> ex: 16 + 18 = 34
> even digit
> even digit

Figure 1: Maud's proof

C_{22}-proofs were always based on a generic example (4/4), and none of them were naive empiricism or crucial experiment. A reason would be that C_{22} is relevant to showing the structure which leads to a generic example and useless for naive empiricism and crucial experiment. However, none of these students could construct a mathematical proof. Three reasons may be offered to understand this. First, the lack of algebraic knowledge as a modelling tool. Second, their proof was always of the type "generic example" and C_{22} is relevant in this case. Third, a lack of method for applying algebra to the expression "2 + 2 + … + 2". In fact, students are not used to an expression like "$a = 2 + 2 + \ldots + 2$ (p terms)". So, we could also suggest that C_{22}-mathematical proof is little probable as a student's proof.

Concerning C_{23}-proofs, while the total number of these proofs is not great (5/37), any of them were not based on naive empiricism, nor crucial experiment, nor generic example. For the former two types of proof, the reason would be same as C_{22}: C_{23} is useless for verifying whether a number is even or not. But why no generic example? We consider that this fact distinguishes the nature of C_{23} from that of C_{22}. The reason would be that algebra as a modelling tool is easy to apply with C_{23}, or the generic example of C_{23} is not enough to persuade for students.

CONCLUSION

In this paper, we have presented a study of the relation between proof and conception, in the case of the problem taken from Healy and Hoyles. In the mathematical analysis of possible proofs, we have shown that the proof character (exhaustion, demonstration evidencing a structure, or another types of proof) would be changed by several conceptions which are possible with a mathematical notion.

In the case study, we got the data of students' proofs and their tendency, and explained them with the results of the theoretical analysis. This would be one response to the results obtained by the study of Healy & Hoyles (1998, 2000). For example, it would be not only the problem of algebra (representation) or of proof conception (proof type) that students could not construct mathematical proof, but also that of conception on a mathematical notion (in our case even number).

For the perspective of mathematical proof learning, our study also showed the necessity to design a situation where students could mobilise more pertinent conception or which enable to shift from a conception to more pertinent one. The mathematical knowledge has a crucial role there.

NOTES

1. This point is similar to the two elements of the formal definition proposed by Balacheff (2000) who models a conception with a quadruplet (P, R, L, ∑) in which: P is a set of problems; R is a set of operators; L is a representation system; ∑ is a control structure.

2. We didn't separate "divided by 2" and "multiplied by 2", because they have the same sub-conceptions.

3. In the French literature, Duval (1995) calls this representation "semiotic register".

4. C_1 is not applicable if numbers are given in the trinary system or odd number system.

5. "Available conception" means in this paper the conception or the operator that is not only known but also which can be used by students.

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