

THE NATURE OF STUDENTS' RULE OF INFERENCE IN PROVING: THE CASE OF REFLECTIVE SYMMETRY

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INTRODUCTION

The aim of this research is the diagnosis of students' state of knowing in the proving process. What is of interest in this paper is the knowing not of proof but of a geometric notion engaged in proving. This is to know *how students construct proof basing on their knowledge*. Many researchers have analyzed the nature of proof and reasoning, its role in mathematics learning and teaching, different type of proof, structure, and so. Some of them have focused on the argumentation (Duval, 1991; Douek, 2000; Pedemonte, 2001; and so). They allow to notice the structure of reasoning especially the rule of inference which can be expressed as a form of implication "if A then B" (see, Duval, 1991; Toulmin, 1958). It appears that the knowledge involved in reasoning comes within the rule of inference, because statements cannot be connected without it.

The questions concerning the rule of inference have been studied, with the term "conditionality of the statements", by the Italian research group (Boero et al., 1996; Boero et al. 1999; etc.). They analyzed the process of its generation, and its link to proving process with relation to the dynamic exploration of the problem situation. The focus of this paper is on the generation process of rule of inference *in the usual proving process in geometry*: which rules of inference and how are they mobilized?

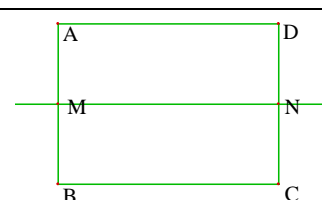
In what follows, observations and case studies are presented from the viewpoint of the rule of inference to make evidence the nature of students' rule of inference.

OBSERVATION

An observation is organized to see how students of 9th grade (aged almost 14 years) in France solve proving problems in reflective symmetry. The task consists of four problems. Just two of them are dealt in this paper.

Problem 2 is to recognize two symmetric segments each other and to prove it. To prove that the segments AD and BC are symmetric, it needs at first that extremities of segments are symmetric, such as "AM = MB and $AB \perp MN$, so A and B are symmetric". After the proof for extremities, it needs the proof for segments.

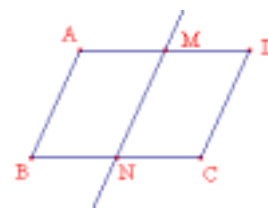
Problem 2: *The quadrilateral ABCD is a rectangle. Let M, N, be the midpoints of opposite sides AB et DC. Are the segments AD et BC symmetric with respect to the line MN? Reply with "yes", "no", or "not always", and prove your answer.*



Problem 3 is very similar to the problem 2. The problem statement is almost same

except that the quadrilateral is at this time parallelogram. In the given figure, the segments AB and DC are not symmetric with respect to the line MN.

Problem 3: The quadrilateral ABCD is a parallelogram. Let M, N, be the midpoints of opposite sides AD et BC. Are the segments AB et DC symmetric with respect to the line MN? Reply with “yes”, “no”, or “not always”, and prove your answer.



AN INCONSISTENCY IN THE ANSWERS

By analyzing the products and protocols for these problems, it is clear that many rules of inference which are not common or correct are mobilized there. Especially, some pairs use the rule of inference for the first problem inconsistent to the second problem from mathematical point of view.

Delphine & Baptiste reply “yes” for the problem 2 and “not always” for the problem 3. For the problem 2, they jointly construct a proof given below (Figure 1). They appear to use the following rule of inference characterizing symmetry in the last sentence (last three lines):

R_1 : “if $Q_1Q_2 \parallel d \parallel Q_3Q_4$ and $Q_1Q_2 = Q_3Q_4$, then $Q_1Q_2 = \text{Sym}(Q_3Q_4, d)$ ”

This rule is not correct. Two segments can be different distance from axis. As they use the perpendicular “ $AM \perp MN$ ” as a hypothesis (while this property is not stated in the problem statement) and the equal distance “ $AM = MB$ ” in the proof, *they have the properties necessary for characterizing symmetry.*

Yes
 $(AM) \perp (MN)$
 $(DN) \perp (MN)$
 $[AM] = [DN]$
 thus $(AD) \parallel (MN)$
 $(BM) \perp (MN)$
 $(NC) \perp (MN)$
 $[BM] = [NC]$
 thus $(BC) \parallel (MN)$
 As the segments $[BC]$ and $[AD]$ are parallel to (MN) and they are the same length, then they are symmetric with respect to the line (MN) .

Figure 1 the transcription of Delphine & Baptiste’s proof for the problem 2

In the protocol, the properties they mentioned for Problem 2 are “perpendicular (right angle)”, “equal distance”, “same length”, “parallel”, and so. Among these properties, two of them are used for the characterization of symmetry. Delphine notices perpendicular and equal distance, and that the former is an important

criterion for the symmetry. More precise analysis for this point is presented in the other paper (Miyakawa, submitted).

In the next problem (Problem 3), Delphine & Baptiste include the perpendicular requirement in arguing that symmetry is absent for the parallelogram case. Their proof is given below (Figure 2). They give correct answer “not always” and point out the absence of a right angle in the given figure. The rule of inference mobilized in Problem 2 is inconsistent to the figure given in Problem 3. In fact, they use R_1 for Problem 2, but the segments (AB and DC) which satisfy this rule in Problem 3 are recognized as non-symmetric segments. The given figure, parallelogram, has two parallel segments to the axis and also the same length. However, the students never mentioned this inconsistency in the protocol.

Not always:

In the reflective symmetry, the segment passing through a point and its symmetric is perpendicular to the symmetry axis.

In the parallelogram ABCD the segment AD isn't perpendicular to the symmetry axis MN.

In the parallelogram ABCD the segment BC isn't perpendicular to the symmetry axis MN.

As the point A isn't symmetric to the point D with respect to the line MN and the point B isn't symmetric to the point C with respect to the line MN so the segment AB isn't symmetric to the line MN.

In the rectangle ABCD which is a particular parallelogram the opposite sides AB and CD are symmetric with respect to the line MN.

Figure 2 the transcription of Delphine & Baptiste's proof for the problem 3

HYPOTHESIS FROM THIS INCONSISTENCY

The question asked from this inconsistency is at first “Why did they mobilize the rule of inference R_1 for the problem 2?” One cannot consider that students accept two statements obviously inconsistent. It brings us to construct the following hypothesis:

(H) “*When the students do not have the rule appropriate for proving, they create themselves some rules from geometric properties already known or identified.*”

This hypothesis can be illustrated as a diagram below (Figure 3). When students are asked to prove a geometric property, they sometimes mobilize some properties for the conditional statement, which are chosen among the properties identified in the given figure. In the case of Delphine & Baptiste, “rectangular”, “midpoints”, “equal distance”, “parallel”, “perpendicular”, and “same distance” are identified. Then, “parallel” and “same distance” are qualified to be conditional statements of the rule

of inference R_1 which characterizes or proves symmetric.

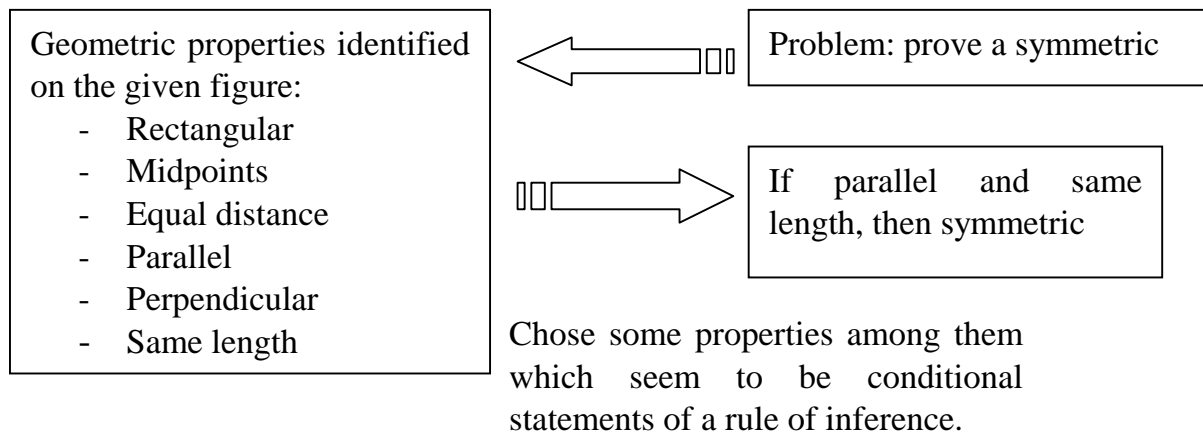


Figure 3 the generation process of a rule of inference in the case of Delphine & Baptiste for the problem 2.

One of arguments for this hypothesis can be seen in the next pair who generates their rule of inference and verifies by construction. The rule of inference they made is not for the reflective symmetry but central symmetry.

GENERATION PROCESS OF RULE OF INFERENCE BY THE STUDENTS

Laura & Justine give the answer “Yes” correctly for the problem 2 and also “Yes” incorrectly for the problem 3. The method of their proving is at first prove symmetry for the extremities and then for the segments. They give almost the same proof for each problem. The proof for the problem 3 is given below (Figure 4). The difference in two proofs is just that for the first, “alignment” is not stated, that is, a conditional statement “alignment” is absent in the rule of inference mobilized in the proof of the problem 2.

Yes
 hyp: ABCD parallelogram
 M milieu de [AD]
 N milieu de [BC]
 Proof: A is the symmetric point of D with respect to M, because it's equidistance from A and D and 3 points are aligned.
 B is the symmetric point of C with respect to N, because it's equidistance from A and D and 3 points are aligned.
 Conclusion: AB and DC are symmetric with respect to the line MN

Figure 4 the transcription of Laura & Justine's proof for problem 3 (my emphasis)

The two rules can be expressed as followings:

“ R_2 : if $PM = MQ$ then $Q = \text{Sym}(P, M)$ ”

“ R_3 : if $PM = MQ$ and P, M Q are aligned then $Q = \text{Sym}(P, M)$ ”

The second rule R_3 is almost correct for the central symmetry¹, but the problem 3 asked the reflective symmetry. This difference between two rules means that they found the necessity of alignment in proving process. Generation process of rule of inference is very clear in the Laura & Justine's protocol. They agree, in the protocol, to write the same proof for the problem 3 as for the problem 2. But Laura asks whether they are really symmetric with the condition of equal distance [167]. And she finds a counter example [169] and draws two points at first, then another point equidistant from two former points (Figure 5). She asks to Justine whether the former two points are symmetric [173]. Then they explicit the necessity of alignment for the central symmetry [174-175].

167. Laura: yes, I don't know, whether a point is equidistant from two points, they are symmetric.

168. Justine: but, yeah, because

169. L: of course ... ah yes, you have equal distance ... look, wait, I draw you a diagram. From A and D, put ... look, if you have a point like this, another, so this is M.

170. J: hum.

171. L: if you have A there, and another point places, but at the equal distance, you see? For example, there,

172. J: ah, yes.

173. L: Is it symmetric?

174. J: oh yeah, it should be that they are ..., how ..., we have to say that they are aligned, three points.

175. L: right, equidistant from A and B, and these three points are aligned. It's what we forgot in the previous proof.

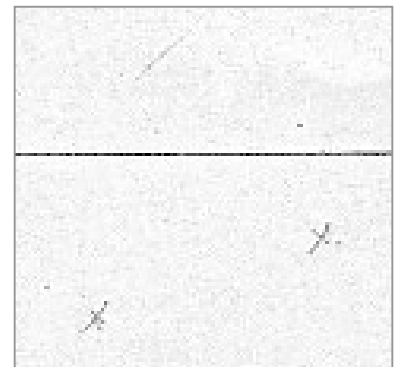


Figure 5 two cross under the horizontal line are A and D, and a light cross above the line is a point M.

Laura adds “alignment” to the conditional statement with the construction of a counter example and generates by herself a rule of inference R_3 . It seems that the construction is a good mean to persuade Justine the necessity of alignment. However, it seems that the rule is used by the students, not because it's admitted or accepted by the theory, but because it allows to construct a figure which is accepted perceptively as a symmetric.

CONCLUDING REMARK AND DISCUSSION

The students themselves generate a rule of inference. The analysis of protocol for two pairs would be examples for evoking and empirically validating the hypothesis (H). On the nature of rule of inference, I propose an educational question conducted from our hypothesis, and finish this paper.

By remarking on the generation process of rule of inference by Laura & Justine, the backing they have for validating the mobilized rule of inference is a construction, such as “if one can construct a figure which is accepted visually or perceptively, these properties can be a conditional statement of the rule of inference”. The construction is a way to validate the rule mobilized. However, a problem is arisen: it is not sure whether the rule validated by construction will be accepted by the theory admitted at the beginning. Mathematically or theoretically, as Duval (1991) states,

¹ “Almost”, because if two points are identical, they are not symmetric with respect to a given point.

the rule of inference should be accepted by the theory admitted at the beginning. But, are the rules of inference mobilized in mathematics always as he says? It seems that we don't pay too much attention if the rule is accepted by the theory or not. This reflection poses an educational question: "What should students' rule of inference rely on?"

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