Understanding cultural aspects of argumentation and proof: A case study of Pythagorean theorem in a Japanese classroom

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The purpose of this study is to reveal the cultural aspects of argumentation and proof in a Japanese classroom introducing Pythagorean theorem to grade 9 students. We delineate a perspective that posits the proof has three aspects, 'structure', 'language', and 'function', to characterise proof and proving in a classroom and better explain its cultural specificities. The results of a case study showed some characteristic features of proof and proving; for instance, the emphasis on the 'if-then' form of the statement to be proven, argumentative structure in the propositional logic, written proof with few ordinary words, the explanatory role of oral proving, and the lack of argumentative activity. Potentials and perspectives towards international comparisons are also discussed.

Keywords: Argumentation, culture, classroom, proof, secondary education.

Introduction

It is difficult for the international community of mathematics education research to reach a consensus about what constitutes proof (e.g., Mariotti et al., 2018) because argumentation and proof are culturally and linguistically shaped. What is called as proof differs according to the countries. For example, in France, *preuve* and *démonstration* are used for the English term proof, but only the latter ('mathematical proof' in English) is accepted within the mathematical community (Balacheff, 1987). While 'proof' and 'mathematical proof' may be considered special cases of 'argumentation' which articulate the reasons supporting the validity of a statement for oneself or somebody, unlike many Western countries, 'argumentation' is an activity rarely found in Japanese society (Sekiguchi & Miyazaki, 2000). Adopting an institutional perspective (Chevallard, 2019), argumentation and proof are objects situated differently in different institutions (e.g., Miyakawa, 2017). Researchers must therefore take into account such cultural specificities to prevent scientific studies on the teaching and learning of proof and proving from only being useful in a specific context (Reid et al., 2022).

This study was part of an international research project on argumentation and proof from linguistic and cultural perspectives. Comparative studies on curriculum documents or textbooks have been conducted within the project (e.g., Hakamata et al., 2022; Otani et al., 2022). In this paper, we reveal the cultural aspects of argumentation and proof in a Japanese classroom. To understand the cultural specificities of teaching and learning proof in a given country, we proposed a perspective to characterise proof-related activities in our earlier study (Miyakawa & Shinno, 2021). The research questions in this study are: *What counts as proof and what counts as proving in a Japanese classroom? What characterises the cultural specificities of proof and proving?* To answer these questions, we illustrated and analysed, in terms of our theoretical perspective, teacher's and students' activities in a Japanese 9th grade mathematics classroom introducing the Pythagorean theorem.

Theoretical framework

Conceptualisations of argumentation and proof in mathematics education have been discussed from different perspectives (Balacheff, 2008; Cabassut et al., 2012; Mariotti et al., 2018; Reid & Knipping, 2010). To shed light on proof and proving in classrooms from a cultural perspective, we propose a model composed of the following three elements (Miyakawa & Shinno, 2021).

- *Structure* refers to the organisation of reasoning or arguments showing how different statements in a proof are connected (e.g., geometric proofs often require a chain of deductive steps, consisting of given statements, a theorem/axiom/definition, and a conclusion).
- *Language* is a semiotic representation (e.g., Duval, 2006), including gestures, oral and/or written discourse, and diagrams, used to express arguments and reasoning structures..
- *Function* denotes the role of proof in different ways (e.g., Hanna, 2000). The function attributed to proof is not reserved for those often mentioned in the literature (e.g., verification, explanation, exploration, communication, or systematisation).

A basic tenet of this model is that 'the triplet allows us to explain that some of the three aspects are emphasised in a given institution more than other aspects, and some aspects are considered in a different way according to the institution' (Miyakawa & Shinno, 2021, p. 256). We apply this model not only to the mathematical proof, but also to the proof-related activities such as argumentation.

A methodological issue with this model is how to specifically analyse each element. While we do not have yet a definite answer, some related concepts or ideas in the previous studies help us to go further. For example, the structure could be elucidated by the *argumentation analysis* with Toulmin's model (Knipping and Reid, 2019). The language can be characterised in different ways, in addition to the overall categories such as ordinary language, mathematical language, naive formalism, and so forth (Balacheff, 1987). The function could be also characterised from different perspectives, such as the concepts of *logical and epistemic values* of a statement (Duval, 1991).

Methods

In the Japanese curriculum, proof is introduced in the geometry domain at grade 8, and different theorems or statements are taught and proven in grades 8–9 in lower secondary schools. Our paper reports the results of the analysis of classroom activities in the case of the Pythagorean theorem, which is taught in grade 9. In curricular documents, national curricula and textbooks, this theorem should be explored (discovered) and proven, and different proofs are expected to be taught.

Classroom video data were collected from a grade 9 mathematics lesson in a middle school affiliated with a national university in June 2022. Due to difficulties seeing students' written proofs in the video, some of their worksheets were collected as additional data.

The analysis consisted of three steps: 1) identifying phases and episodes from the classroom video, 2) creating transcripts of each episode, and 3) analysing the transcripts and worksheets in terms of the theoretical model. In the first step, we worked on the classroom video to explore and identify moments that included proof-related activities. In this study, we focus on introducing, formulating, and proving the Pythagorean theorem.

Analysis and results

Lesson structure

Figure 1 shows the structure of the lesson which consists of five phases in terms of the classroom activities and (approximate) time allocation.

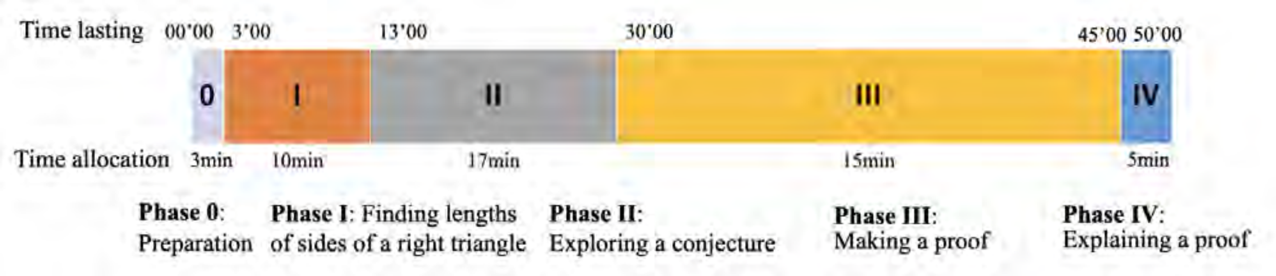

Figure 1: Lesson structure

Students mostly worked individually in Phases I and II but occasionally interacted with their seatmates. Phases III and IV were collective activities carried out in groups. Phase III was the time for constructing a proof. Phase IV was devoted to the exchange between groups, in which students within a group explained their proof to students of other groups. In the following section, we briefly describe the activities in Phases I and II, and then focus on those in Phases III and IV to analyse the proof in terms of its structure, language, and function.

Classroom activities in Phases I and II

The lesson begins with the teacher proposing to create right triangles on GeoGebra (for iPad) and then finding a set of the lengths of three sides. The teacher prepared material on GeoGebra which allowed students to automatically measure the length of the sides. After approximately 10 minutes, the teacher wrote down students' findings (e.g., [5, 12, 13]; [6, 8, 10]) on the board. The teacher then asked them to look for a conjecture (or pattern) which might be true on the five or six triplet numbers they had found, but nobody mentioned conjectures related to the Pythagorean theorem. One student presented an idea in the case of irrational numbers, but the teacher did not go into detail about it.

Introduction of the theorem and the proving task

The following utterance shows how the teacher introduced the Pythagorean theorem in class.

24:38 Teacher: Okay ... well, I do not think anyone wrote this down, for *a*, *b*, and *c*, an ancient man discovered that when squaring *a*, *b*, and *c* respectively, adding them [a^2 and b^2] provides a number equal to the square of *c*. He discovered this. How about this, everyone? Try it. Make sure that if you add squared and squared, do you get the square of *c*? Really? […]

After introducing the expression '$a^2+b^2=c^2$', the teacher asked the students to verify it in several cases using a calculator (for the cases of irrationals, the teacher asked for approximate numbers to be used). Following this activity, the teacher formulated the theorem and asked the students to prove it. The next episode shows what proving is for and how students' proving activities are organised.

27:49 Teacher: Well, you think it is probably true. The ancient man in mathematics discovered it, and it had a general name. You may know the name. The Three-square theorem [Pythagorean theorem] is used because it has three

	squares. […] This means that if triangle ABC is a right triangle. It cannot be an equilateral triangle. If it is a right triangle, then the theorem states that only if it is, then when we square it up, we obtain this relationship. This is the Three-squares theorem. I do not think anyone wrote this down, but it would be great if you could work it out for yourself.
30:06 Teacher:	Well. What would you like to do if you were told it was a theorem? What do you want to do? Well, when you say a theorem, you want to prove it. I want you to prove it, to see if it is really true. You want to prove it, don't you?
30:49 Teacher:	Well. As far as I know, there are more than hundred ways of proving. Pythagorean theorem is… actually, we have done it when you were 7th graders. But you don't remember it. I remember it, but... You do not [...].
31:25 Teacher:	So today we will focus on two of these methods. One is to prove it using this diagram [Figure 2]. The other uses this diagram [Figure 3]. Now you'll work on the proof in a group. You will know later which figure will be given. Look forward to it. You will have one or the other. Two students in a 4-people group will use one of the diagrams. If you considered the one proof, then have time to explain your proof to someone who considered another proof. So, to fully understand the proof, please try to think about it carefully in the remaining time. Okay, let us form your groups.

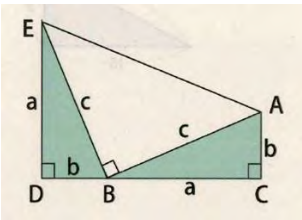

Figure 2: Diagram A

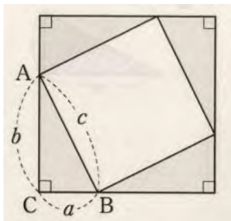

Figure 3: Diagram B

After explaining the meaning of the theorem, the teacher provided two diagrams (Figures 2 and 3) which implied two ways of proving. The teacher asked the students to work on one of the diagrams and write a proof on their worksheets as a group. The task given in the worksheet was '*Prove that Pythagorean theorem is true by using the following diagram*'. Notably, the teacher did not write the task on the board as '*Prove…*' but as '*Consider a proof of Pythagorean theorem*'. 'Consider a proof…' implies that students must find an already existing proof. The teacher provided the diagrams without any information about how they were constructed except for a few signs and symbols to represent equal or right-angled relationships; for instance, in diagram A, a quadrilateral ACDE is a trapezoid, two triangles ABC and BED are congruent, and so forth. Therefore, the students were required to assume these properties for proving.

The teacher also created an opportunity for the students to explain and exchange their proofs with other groups in Phase IV. Through this activity, it seems that the teacher expected the students to gain a better understanding of the proof when working in groups, as they would have to exchange their proofs with other students.

Examples of written proofs

Many students created proofs written in algebraic expressions (e.g., Figure 4 for diagram A and Figure 5 for diagram B). In these proofs, only a few ordinary Japanese words were included (For Proof B1, no Japanese language was used). Ordinary words were mostly used to refer to figures and/or describe algebraic expressions or transformations.

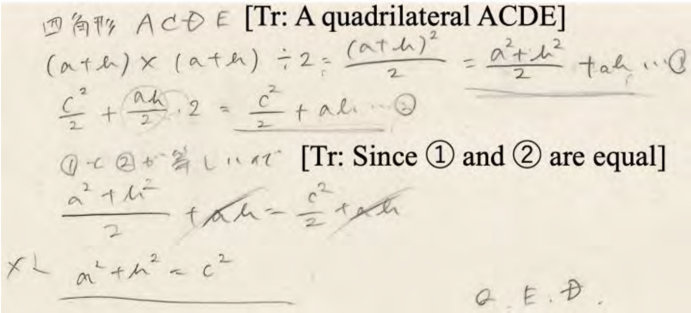

Figure 4: Proof A1

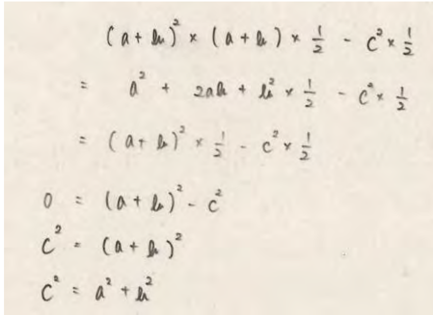

Figure 5: Proof B1

From the perspectives of structure, language, and function

Structure. The structure of the students' proof can be reconstructed as shown in Figure 6 in the case of Proof A1 (Figure 4). This is a result of an analysis based on the argumentative structure proposed in previous studies (Knipping & Reid, 2019). Some elements in Figure 6 are given in dotted line boxes because they are implicit in the student's proof. The hypothesis (△ABC is a right triangle) is included as a starting statement, given in the task. In addition, the given diagram (Figure 2) itself is an element of Figure 6 because the proof assumes several conditions and properties not explicitly mentioned. Although Figure 6 only illustrated one case, a common structure was found in the proofs produced by other students.

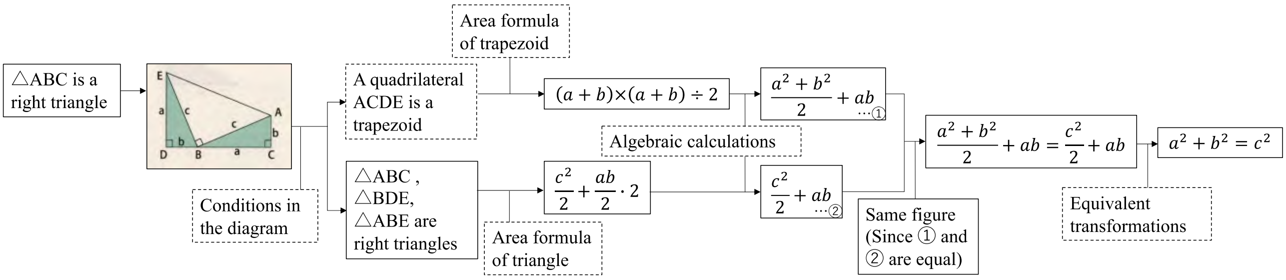

Figure 6: An argumentative structure of the proof on diagram A (Proof A1 in Figure 4)

One characteristic of the structure is that students' proofs are based on propositional logic. This was also evident in the classroom discussion where any quantification (such as 'all' or 'every') used in the predicate logic was never mentioned. Further, as the teacher emphasised the theorem in an 'if-then' form when it was being formulated (the hypothesis is 'if triangle ABC is a right triangle' and the conclusion is 'the statement $a^2+b^2=c^2$ is true'), the proof is considered to have a structure of propositional logic that connects the hypothesis to the conclusion.

Language. Students' written proofs on the worksheets are given in algebraic form without explanation in the Japanese language (see Figures 3–4). Looking at the ordinary words used in their proofs, no sentences with subject or verb were used. In their written proofs (Figure 6), each step in a logical chain is often implicit. The reasoning structure supporting algebraic expressions or transformations are rarely written explicitly. Students' proofs heavily relied on the given diagrams. The geometric properties in the diagrams, which were often implicit in their writings, played different roles (conditions, properties, etc.) essential for formulating the algebraic expressions and their transformations (see Figure 6). Some students drew on the diagram to show the relationships (such as 'markings' for equal segments and congruent triangles) but did not express them as texts.

Regarding the proving activities during the groupwork, it is difficult to distinguish spoken and written arguments. It is unclear how these different modalities were incorporated into their productions because the proving activities were performed collectively and dialogically. However, we can see the importance of oral and gestural arguments when they convey their proofs to others. Although their written proofs in the worksheets involved many undescribed or implicit aspects, some of these aspects might have been mentioned during discussions with other students. Ordinary language supporting warrants or reasoning structures could be used as auxiliary words for their proofs. What counts as 'proof' in the classroom is not necessarily formulated as a written entity, but is often provided orally.

Function. When the teacher asked for a proof, he said, 'I want you to prove it, *to see if it is really true*. You want to prove it, don't you?' It seems that the proof, at least for the teacher, plays a role in verifying the truth of the theorem. In other words, the proof provides the theorem with a logical value (Duval, 1991), allowing the students to reuse it in other (proving) tasks. By contrast, we could not see the function of changing the epistemic value of statement for someone else (Duval, 1991). Before coming up with the proof, students already knew that the theorem is true since the teacher introduced it as a property that was already discovered by ancient people, and since the students have already verified the proof with or without using calculators in some cases.

Further, in this lesson, it seems that the proof itself does not have explanatory power without complementary information (diagrams, explanations, gestures, etc.). The point here is that the proof has already been given as a product of the groupwork, and the students had to explain it. The proof (at least for the students) was not used to convince someone but was an object to be *explained to others*. 'To prove the theorem' means 'to get the logical structure' which allows students to see how the conclusion can be derived from the hypothesis with the chain of reasoning. Logical structure is something that students can explain to others. To this end, the students first constructed the proofs and then needed to *understand* and explained them, as the teacher encouraged them to do so.

Discussion and conclusion

Let us answer and discuss the three research questions in terms of a triplet analysis.

Answer to RQ1: What counts as proof in a Japanese classroom? Proof can be explained orally and comprehensively. A written proof does not necessarily have explanatory power without oral and/or gestural arguments. The logical structure constructed as a result of the proving phase is considered to be proof. This is why provers try to convey it to other people not only through a written proof, but also through oral and gestural communication.

Answer to RQ2: What counts as proving in a Japanese classroom? The proof is performed after verifying that the theorem is true in some cases. There is no room to doubt the truth of the statement before proving it. This is the process of creating a path of reasoning from the given hypotheses to the conclusion. For this reason, the teacher and textbook emphasise stating the theorem in an if-then form in the beginning ('if triangle ABC is a right triangle, then $a^2+b^2=c^2$'). Proving may be a collective (group) activity and is not an act of argumentation to convince or persuade someone.

Answer to RQ3: What characterises the cultural specificities of proof and proving? The cultural specificities mentioned in the RQ1 and RQ2 should be understood at different levels. For example, we identified verification as a function of proving in the observed lesson. In other lessons, the same teacher's questioning prompted students' proving activities to help them understand why a theorem is true, which is associated with the function of explanation or illumination (Hanna, 2000). Thus, there are differences even within Japanese classrooms at a certain level. However, there are 'common' cultural aspects we have identified in our earlier study: for instance, the emphasis on the if-then structure in propositional logic, the explanatory role of oral proving, and the lack of argumentative activity to convince someone. The triplet allowed us to explain that these aspects are more emphasised than other aspects in the Japanese classroom, at least in our empirical data.

Some earlier studies on proof and proving in the Japanese classroom have also investigated some of the three aspects, but not from a cultural perspective. In those studies, the cultural specificities of Japanese classrooms, although there is a discussion by Sekiguchi and Miyazaki (2000), are rarely studied empirically. In this regard, our study would be a new contribution not only to a deeper understanding of the Japanese classroom but also to the cultural approach for investigating proof and proving. The triplet model can be extended to international comparative research (e.g., Hakamata et al., 2022). For example, proof and proving in a German classroom regarding the teaching and learning of Pythagorean theorem (described by Knipping [2004] as *intuitive-visual argumentation*) is similar to the Japanese case, while it contrasts with a French case in which the importance of discursive argumentation, where each step of a logical chain should be written explicitly, can be observed (Knipping, 2004). Therefore, further research is needed on international comparisons to develop a method to distinguish which aspects are considered 'specific' within and across countries.

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References

Balacheff, N. (1987). Processus de preuves et situations de validation. *Educational Studies in Mathematics*, *18*(2), 147–176. https://doi.org/10.1007/BF00314724

Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: An essay on the case of proof. *ZDM - Mathematics Education*, *40*, 501–512. https://doi.org/10.1007/s11858-008-0103-2

Cabassut, R., Conner, A., İşçimen, F. A. Furinghetti, F., Jahnke, H. N., & Morselli, F. (2012). Conceptions of proof – In research and teaching. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education: The 19th ICMI study* (pp. 169–190). Springer. https://doi.org/10.1007/978-94-007-2129-6_7

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114. https://doi.org/10.24529/hjme.1205

Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics*, *22*(3), 233–261. https://doi.org/10.1007/BF00368340

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*, 103–131. https://doi.org/10.1007/s10649-006-0400-z

Hakamata, R., Bredow, F., Knipping, C., Miyakawa, T., & Shinno, Y. (2022). Comparing the usage of proof-related words in German and Japanese mathematics textbooks. In C. Fernández et al. (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, p. 221). PME.

Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics*, *44*, 5–23. https://doi.org/10.1023/A:1012737223465

Knipping, C. (2004) Argumentations in proving discourses in mathematics classrooms. In G. Törner et al. (Ed.). *Developments in mathematics education in German-speaking countries*. (pp. 73–84). Ludwigsburg, Verlag Franzbecker.

Knipping, C., & Reid, D. (2019). Argumentation analyses for early career researchers. In G. Kaiser, & N. Presmeg (Eds.). *Compendium for Early Career Researchers in Mathematics Education* (pp. 3–31). Springer. https://doi.org/10.1007/978-3-030-15636-7_1

Mariotti, A. M., Durand-Guerrier, V., & Stylianides, G. (2018). Argumentation and proof. In T. Dreyfus et al. (Eds.). *Developing research in mathematics education: Twenty years of communication, cooperation and collaboration in Europe* (pp. 75–89). Routledge.

Miyakawa, T. (2017). Comparative analysis on the nature of proof to be taught in geometry: The cases of French and Japanese lower secondary schools. *Educational Studies in Mathematics*, *94* (1), 37–54. https://doi.org/10.1007/s10649-016-9711-x

Miyakawa, T. & Shinno, Y. (2021). Characterizing proof and proving in the classroom from a cultural perspective. In M. Inprasitha et al. (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 250–257). PME.

Otani, H., Reid, D., & Shinno, Y. (2022). How are proof and proving conceptualized in mathematics curriculum documents in the USA and Japan? In C. Fernández et al. (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 267–274). PME.

Reid, D. A. & Knipping, C. (2010). *Proof in mathematics education*. Sense Publishers.

Reid, D., Shinno, Y, & Fujita, T. (2022). International perspective on proof and proving: Recent results and future directions. In C. Fernández. et al. (Eds.). *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 211–212). PME.

Sekiguchi, Y. & Miyazaki, M. (2000). Argumentation and mathematical proof in Japan. *The Proof Newsletter*.