CHARACTERIZING PROOF AND PROVING IN THE CLASSROOM FROM A CULTURAL PERSPECTIVE

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This theoretical paper proposes a new perspective on identifying and characterizing the cultural specificities of proof and proving in the classrooms of a given country. To this end, based on the related literature, researchers propose "structure", "language", and "function" as a triplet of aspects that constitute proving activities. Researchers then exemplify each aspect in an example case of proving activities in a Japanese classroom and discuss how it allows us to characterize the cultural specificities of proof and proving.

INTRODUCTION

What is this thing called "proof"? The meaning of proof and proving is still a subject of debate among researchers in mathematics education (e.g., Stylianides, Bieda, & Morselli, 2016). Some previous studies have shown how proof and proving are differently situated in the curricula, textbooks, and classroom practices of different countries (e.g., Jones & Fujita, 2013; Knipping, 2004; Miyakawa, 2017). These studies imply that proof and proving in mathematics are culturally embedded activities. However, the number of earlier studies on proof and proving from a cultural or international perspective is relatively small (Reid, Jones, & Even, 2019); therefore, further theoretical, and empirical studies on this topic are needed.

Since the diversity in the definitions of proof and proving may constitute an obstacle to international communication among researchers from different countries (Reid, 2015), it is important to take into account researcher's epistemology on proof (Balacheff, 2008) as well as the cultural dimension of proof (e.g., Shinno et al., 2018). Although the expression "cultural" is rather ambiguous, researchers understand it as "institutional", based on *The Anthropological Theory of the Didactic* (hereafter the ATD; Chevallard, 2019). Within the ATD, a mathematical object (proof and proving in our case) exists in each *institution* under the influence of several factors of different origins. According to this tenet, cultural factors determine the way in which students relate to proof and proving and produce diversity according to the institution to which the object belongs. To gain deeper insight into the cultural specificities of proof and proving in a given institution, researchers need to develop a theoretical lens to analyze and explain such specificities. Therefore, the research question in this research is as follows: *What are the critical aspects that allow researchers to identify the cultural or institutional specificities of proof and proving in the mathematics classrooms of a given country?* To answer this, we propose a new theoretical perspective and then exemplify it in an example case of proving activities in a Japanese middle school classroom.

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THEORETICAL PERSPECTIVE

Related Literature: Different Aspects of Proof and Proving

What counts as a principal aspect of proof depends on the theoretical perspective adopted for the research being undertaken. Mariotti et al. (1997) proposed the notion of *mathematical theorem* that consists of a system of relations between a *statement*, its *proof*, and the *theory* within which the proof makes sense. Balacheff (1987) proposed a framework composed of *knowledge*, *formulation*, and *validation*. These works certainly deal with crucial aspects when analyzing and understanding the complex nature of proof and proving, especially in relation to the mathematical knowledge behind it.

When discussing proof and proving in the classroom, another important aspect is its relationship with argumentation, which directs us to the discourse or rhetorical means to convince others (Stylianides et al., 2016, p.316). Duval (1991) nicely characterized the functional aspect of mathematical proof with respect to argumentation in terms of *the epistemic* and *logical values* attributed to the statement to be proved: The mathematical proof provides the logical value (true or false), while the argumentation changes the epistemic value that is the degree of certainty the collocutor has with the statement (certainly, probably, etc.). Knipping (2008) adopted Toulmin's argument model to describe the argumentation structure in the proving process. This offers the *argumentation analyses* that make it possible to compare and infer the rationale of the argumentation in both local and global structures through classroom talk (Knipping & Reid, 2013). It is also important to note that argumentation is the act of persuading someone with a claim in ordinary life, since proving often plays this role even though it is not an ordinary practice in some countries (Sekiguchi, 2002; Sekiguchi & Miyazaki, 2000).

Although different theoretical approaches have been used to characterize proof and proving (or argumentation) in mathematics education so far, cultural, or institutional perspectives are rarely considered. Nevertheless, some ideas from the existing frameworks mentioned above can be reconsidered and integrated into a new theoretical perspective to characterize what constitutes proof and proving from a cultural perspective. Our proposition is that proving activities in a given institution can be characterized by three aspects: *Structure*, *language*, and *function*. Although these aspects have already been mentioned in different ways in previous studies, researchers intend to reconceptualize them as a triplet to identify the cultural specificities of proving in the classroom. Let us briefly explain each aspect, and then provide some examples.

Proof and Proving as a Triplet: *Structure*, *Language*, and *Function*

Structure denotes here the organization of reasoning or arguments showing how different statements consisting a proof are connected. The structure required in a proof may differ according to the institution. For example, in our everyday life, arguments are given one after another to persuade the collocutor of the validity of a statement (Duval, 1991). In contrast, proof and proving in school geometry often requires a basic step consisting of "given statements", a "theorem/axiom/definition", and a "conclusion", as well as the chain of steps in the propositional logic. At a more advanced level, it may be more appropriate to grasp the reasoning for the universal proposition in the predicate logic including quantification

(Durand-Guerrier & Arsac, 2005). The argumentation analysis (Knipping, 2008) is a way to identify such a structure that may be implicit in classroom interactions.

Language is the semiotic representation, *register* or not (Duval, 2006), used in a given institution to express the arguments and structure of reasoning. Gestures and oral discursive representations are also languages that may express arguments and the structure of proof. In the classroom, different representations are used such as gestures, oral and/or written discourse, diagrams, and so forth (Chen & Herbst, 2013). Since the formulation of proof is concerned with ordinary language, this aspect shows strong cultural effects at both the grammatical and semantic levels.

Function has been extensively studied in the mathematics education research field (e.g., Hanna, 2000). However, the function attributed to the proof differs according to the institution and is not reserved to the ones often mentioned in the literature (verification, illumination, communication, systematization, and discovery). In other words, what is called "proof" may differ according to the function attributed to the justification. For example, Miyakawa (2017) showed that in French lower secondary schools, proof is a means to justify a statement without relying on perception or visual information.

EXAMPLES: PROVING IN JAPANESE LESSONS

To exemplify how and to what extent our theoretical perspective allows us to account for the cultural specificities of proof and proving in the classroom, researchers will provide empirical data to be analyzed in the next section.

Lessons in a Japanese Middle School

Researchers collected data on ordinary mathematics lessons in a public middle school. A series of five mathematics lessons given to a Grade 8 class (13–14 years old; 35–40 students) were videotaped. A teacher was asked to give ordinary lessons for a unit on "conditions for a parallelogram", right after teaching proofs and congruent triangles in geometry lessons. This middle school teacher had around 20 years of teaching experience.

One of the specificities in the proving activities through the five lessons is that the written proof was given only twice, while there were 10 oral proving activities for the true statements (and five others for the false statements). Below, researchers describe two example cases of proving: One given orally, and another given orally then in a written form.

Case 1: Oral Proving

Case 1 is taken from the third lesson, in which the class proved one by one different statements on the conditions for a parallelogram. The statement was that a quadrilateral with two pairs of equal opposite sides is a parallelogram, which is considered as an important theorem in the textbook used in this class. The teacher reviewed the proof of the statement as follows:

Teacher (T): I will tell how we proved it. Pay attention. Well, by drawing [a diagonal], two triangles appear. A pair of sides are equal. The second pair, [are] equal, the third pair, overlapping ones are equal [T draws a round mark on the diagonal (Figure 1)], so [they are] congruent. Since [they are] congruent, this angle and this angle are equal [T draws marks for equal angles]. Well, since [they are] congruent, this angle and this angle are equal [T draws marks for equal angles]. Well, next. Since this angle and this angle are equal, Z appears, and [they are] alternate-interior angles [T draws marks for parallel lines], [they are] parallel. Well, [they are] parallel. Then, here, and here, since the angles are equal, Z appears, and [they are] alternate-interior angles [T draws mark for parallel lines]. Well, it [the diagram] is now a mess, but we can say [it is] a parallelogram. Okay, we can say it.

Fig. 1: Oral proving by the teacher

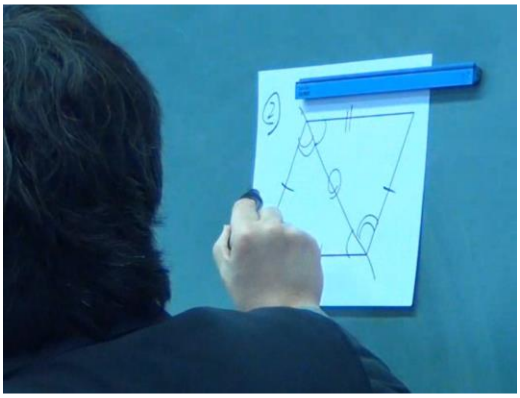
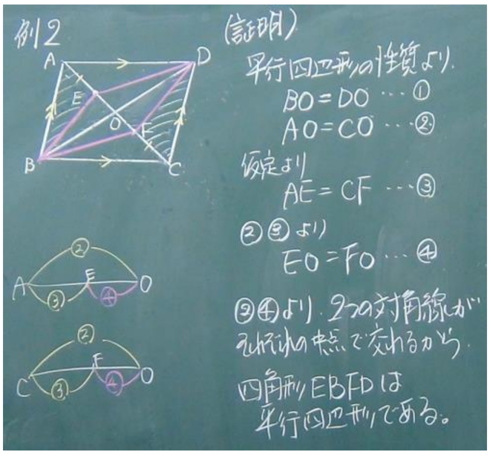


Figure 1: Oral proving by the teacher

Case 2: Written Proof

In the fourth lesson, the teacher proved the statement "Like the right diagram, when taking two points E and F so that AE = CF on the diagonal AC of the parallelogram ABCD, prove that the quadrilateral EBFD is a parallelogram," which is an exercise in the textbook. The statement and proof were first given orally through interactions with students, and then the proof was given in written form on the blackboard (Figure 2). The proving process was therefore divided into two phases: Oral proving and its formulation as a written proof, as in the following transcript:



(Proof)
Based on the property of a parallelogram,
BO = DO ... (1)
AO = CO ... (2)
From hypothesis
AE = CF ... (3)
From (2) and (3)
EO = FO ... (4)
From (2) and (4), since the two diagonals intersect at their midpoints, the quadrilateral EBFD is a parallelogram.

Figure 2: A written proof given on the black board and its English translation

Teacher: We are going to write what I said now. At first, we say that BO and DO are equal, and AO and CO are equal. This is like what we have done so

> far. This is not new. Well, as the quadrilateral at the very beginning is a parallelogram, what we can use is this [writing on the board]. Ok? Well, [it is] the property of parallelogram.

EXEMPLIFYING PROOF AND PROVING AS A TRIPLET

Researchers will now analyze these two cases together from our theoretical perspective of proof and proving as a triplet, by identify the *structure* of reasoning given in these cases, the *language* used to describe the structure, and the *function* played by proof and proving in the classroom.

Structure of Reasoning

The proving in Case 1 consists of the explanations of the congruent triangles and parallel lines by the alternate-interior angles. One could identify the structure of reasoning from the teacher's speech and gestures on the diagram in the video (whereas there are many implicit points): The first step is to prove that the three pairs of sides of the two triangles are respectively equal; the second is to prove the congruent triangles; the third is to prove the equal angles; the fourth is to prove the parallel lines, and the last is to prove the parallelogram. This structure is based on propositional logic, since the teacher never mentions quantifications such as "any" or "all", and the statements are connected from the hypotheses to the conclusion by means of the geometrical properties (definition and theorems).

In Case 2, the class is seeking the reasoning structure that connects the hypotheses to the conclusion. The process of proving goes backward from the conclusion to the hypotheses. This is different from Case 1, in which the teacher proved forward like a written proof from hypothesis to the conclusion. However, the reasoning structure one may identify in the oral proving and written proof is similar to Case 1. The statements are connected by the properties in each step, and the intermediate conclusion is reused in the next step as a given statement until the target conclusion. This structure of reasoning is more explicit in the written proof. In addition, quantification is not mentioned in this case either.

Language

What kinds of language are used to express the structure of reasoning? In the oral proving of both cases, the language used by the teacher is an amalgam of oral discourse, diagrams, and gestures. What he says could not be a written proof per se and would not make sense without gestures and diagrams (see Case 1). Further, in Case 1, the teacher never uses labels (e.g., A, B, C, etc.) when referring to the points or angles. Instead, he often uses "demonstrative words" (e.g., "this" or "they"). This implies that the teacher sees the diagram and gestures as parts of the proving.

In Case 2, the teacher provides a written proof after the oral proving (Figure 2). It is given in Japanese with many symbols. However, the proof consists of a list of symbolized statements with properties, but not of proper Japanese sentences, except for the last line, because each statement does not include a subject and verb and there is no *dot* at the end of the line, which there should be in proper sentences. Unlike English, expressions like "BO = DO" are not abbreviated sentences (as in Nesselmann's *Syncopated algebra*) in Japanese, but symbolic statements that do not preserve Japanese grammar (since the verb should come at the end of the sentence). The teacher says when writing the last line, “The conclusion must be written in Japanese. This becomes a little bit long”. This utterance supports our interpretation that the other lines are not given in proper Japanese.

Function of Proving

One may identify some functions of proving in the two cases. In Case 1, the proving is to provide a *logical value* (Duval, 1991), allowing them to reuse that statement in other proofs, and to systematize geometrical knowledge that has been learned in the previous grades (e.g., parallelograms in primary school). Notably, this was not done to convince someone. The students already knew this statement from the previous lesson and there was no discussion on its truth. One cannot see, therefore, the function of argumentation that changes a student’s *epistemic value* (Duval, 1991) of the statement, and there was no argumentative activity in the observed lessons. This was also the case in Case 2. The class never discussed the truth of the given statement but took it for granted. In fact, the teacher introduced the statement after drawing the diagram, as follows:

Teacher: Well, this is the problem to prove that the red quadrilateral becomes a parallelogram. It looks like a parallelogram, this red one. It also looks like a rhombus. But this becomes a parallelogram. Why is it? This is a problem.

One may see here that another function of the proof is to explain why the statement holds. This role was accomplished by orally exposing the structure of reasoning that connects the hypotheses to the conclusion. In other words, the main function of proving was to create a deductive chain reaching to the conclusion from the hypotheses and show the logical structure. The teacher conveyed this logical structure through oral discursive language with diagrams and gestures.

DISCUSSION

What is Proof and Proving in Japanese Middle School?

One could find in both cases that the teacher did not accord importance to the written proof. It was given only twice in the sequence of five lessons. This implies that what counts as “proof” in this classroom is not necessarily formulated as a written entity. Instead, the oral proving with diagrams and gestures was considered as a proof. This may cause some ambiguities in the connections between statements. One may not explicitly see how the hypotheses are connected to the conclusion by means of a theorem, because the discursive language is not well organized and the “if-then” form is not used to describe the properties. Further, even in the written proof, although it complements some logical connections of statements in the oral proving, there are still some ambiguities due to the non-use of the “if-then” form and the Japanese language.

So, what is the basis to conclude that a statement has been proved in a Japanese classroom? The proving activity is not an argumentative activity to convince someone of the truth of a statement (this is aligned with Sekiguchi’s [2002] claim), but an activity to establish the structure of deductive reasoning from the hypotheses to the conclusion in the propositional logic with a specific level of linguistic rigor (as described above) and to explain why the statement is considered to be true. Further, the proven statement should be universal, as mentioned by Miyakawa (2017), but quantification is not explicitly dealt with in the structure of reasoning.

Theoretical Reflection: Proof and Proving as a Triplet

Let us reflect on our theoretical perspective, conceptualizing proof as a triplet, to answer our research question. This perspective allows us to characterize proof-related activities, such as oral argumentation and written mathematical proofs, from a broader perspective than the existing frameworks. However, we do not intend to argue that these three aspects are necessary and sufficient for discussing the cultural specificities of proof and proving. Instead, researchers claim, based on the results of previous international comparative studies, that *the triplet allows us to explain that some of the three aspects are emphasized in a given institution more than other aspects, and some aspects are considered in a different way according to the institution*.

For instance, in a comparative study of French and German classrooms, Knipping (2004) identified the importance of writing in French classrooms, where each step in an argumentation chain should be made explicit. One may see in our analysis that the linguistic level of rigor required for proving in France is very different from that in Japan. Proving in a Japanese classroom is rather similar to that in a German classroom, which is called *intuitive-visual argumentation*. This example implies the relevance of our perspective that language is an aspect that strongly reflects cultural specificities, in addition to the function of proving mentioned above.

From the institutional perspective of ATD, students' relation to proof and proving could be different not only across the country, but also across grades and mathematical domains (e.g., algebra and geometry) within a country. While in this paper researchers consider as an institution the geometry lessons in a specific country, it is necessary to further analyze and compare empirical data from different institutions to corroborate our theoretical perspective and elaborate it to systematically investigate different proving activities by means of the triplet—structure, language, and function.

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