CHALLENGES IN CURRICULUM DEVELOPMENT FOR MATHEMATICAL PROOF IN SECONDARY SCHOOL: CULTURAL DIMENSIONS TO BE CONSIDERED

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> Depuis les Grecs, qui dit Mathématique, dit demonstration (Bourbaki, 1954).

Within the Japanese mathematics education community, we use two terms for proof: *shōmei* (証明) and *ronshō* (論証). Both terms share the same Chinese character, *shō* 証. Literally, *shō* 証 means 'evidence' or 'proof', *mei*明 means 'clarity' or 'shining', and *ron*論 means 'argument' or 'logic' [1]. Sometimes *ronshō* is seen as a special type of *shōmei*. For example, the current categories of research presentations at the annual conference of the Japan Society of Mathematical Education (JSME) include the category "*shōmei*, including *ronshō* and *setsumei*" (*setsumei* means 'explanation') [2]. Starting from this distinction, we explore cultural and linguistic issues related to teaching proof in Japan and an epistemological perspective on what constitutes proof.

The difference between *shōmei* and *ronshō* is similar, but not identical to the distinction between *preuve* and *démonstration* in French. Balacheff uses *preuve* and *démonstration* in the following way:

> We call proof an explanation accepted by a given community at a given moment… Within the mathematical community only explanations adopting a particular form can be accepted as proofs. They are an organized succession of statements following specified rules: a statement is known to be true or is deduced from those which precede it using a deductive rule taken from a well defined set of rules. We call such proofs "démonstrations". (Balacheff, 1987, p. 148; English translation from Reid & Knipping, 2010, pp. 32–33)

In English we find the expression 'demonstrative geometry' in texts from the early twentieth century (*e.g.,* Smith, 1919; Taylor, 1930). Japanese translations use *ronshō kika* for 'demonstrative geometry'; *kika* means 'geometry'. *Shōmei kika* is never used as a translation. As 'demonstration' is rarely used to refer to mathematical proofs these days (Reid & Knipping, 2010, p. 32), and as Balacheff translates *démonstration* as 'mathematical proof' we will use the same phrase here, without intending to imply that *ronshō* means exactly what Balacheff means by *démonstration* or what an English speaker might mean by 'mathematical proof'. We will translate *shōmei* as 'proof', intending to capture its broader meaning. Already in attempting to specify the topic of our article, we have evidence for cultural and linguistic differences related to proof.

We believe that such cultural and linguistic differences may affect curriculum development in different countries. Our aim here is thus two-fold. First, we wish to reveal the cultural and linguistic issues that need to be considered in the development of curricular content and sequencing for teaching mathematical proof in secondary schools in Japan. Second, we seek to elaborate an epistemological perspective that may allow us to understand what constitutes proof in the curriculum of a given country, so that our results may be applied in international contexts. This second aim can be achieved by looking back at cultural discussions on Japan and by looking ahead to further research opportunities in the international context.

Language and the triplet which composes the Mathematical Theorem

Let us discuss a bit more the two Japanese terms *shōmei* and *ronshō*. Some Japanese works distinguish the word *ronshō* from *shōmei* by referring to their relationship with the system of mathematics. For example, in a book published for Japanese mathematics teachers in the New Math era (JSME, 1966), *shōmei* was described as "mainly related to the truth or falsity of a proposition", whereas *ronshō* was described as "mainly related to the truth or falsity of *a set of* propositions" (p. 366, emphasis added). Some Japanese researchers (e.g., Hirabayashi, 1991; Iwasaki, 1985; Minato, 1974) have also advocated a similar distinction, such that *shōmei* is related to deriving consequences from premises for establishing the truth of a proposition, while *ronshō* is related to the (axiomatic) system in which logical relations between propositions take place.

These differences in the use of the two words *shōmei* and *ronshō* reveal something about the Japanese mathematics curriculum. That is, the use of *ronshō* indicates an emphasis on a systematic approach to mathematics. This is probably because the content and sequencing of the Japanese curriculum in geometry bears a strong resemblance to that of the geometry in Euclid's *Elements*. Miyakawa (2017) pointed out that the Japanese meaning of *shōmei* is reserved

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particularly for general statements and includes constructing the system of geometry. The word *ronshō* describes more precisely constructing the system of geometry but is not used in Japanese textbooks.

Antonini & Mariotti (2008) state "it is not possible to grasp the sense of a *mathematical proof* without linking it to the other two elements: a *statement*, that the proof provides a support and *a theory, i.e.* the theoretical frame within which this support makes sense." (p. 403). According to this model a 'Mathematical Theorem' is a system of statement, proof and theory. The third element, theory, can be considered as a certain mathematical context that provides an underlying basis for a mathematical proof within a domain that can be geometrical, arithmetic, algebraic, and so on. In these terms the distinction between *shōmei* and *ronshō* is at the level of the theory involved. Hanna & Jahnke (2002) distinguish between 'small' and 'large' theories. They give axiomatic Euclidean geometry as an example of a large theory, suggesting that one way to make sense of *ronshō* is as proof in such a large theory. A proof in a small theory, locally organized and perhaps referring to physical mechanisms, seems more like *shōmei*.

This difference, at the level of theory, is epistemological rather than linguistic. We now consider some linguistic issues related to statement and proof in Japanese, and then return to the epistemological level of theory.

Linguistic issues: universal quantification

When we turn to the linguistic aspects of statement and proof, we must consider the influence of natural language used in daily life on the teaching and learning of mathematical proof. Concerning this critical point, Mejia-Ramos and Inglis (2011) investigated the semantic contamination of mathematical proof, where the meaning of a term associated with proof used in natural language influences how the term is understood by students in the mathematical discourse. Although their study focused on the English usage of the term 'proof' as both a noun and a verb, we will examine not only the usage of the term but also the form of the statement to be proven in a certain mathematical domain. In addressing the linguistic issues faced with respect to statement and proof, different formulations of statement and multiple meanings of proof can be identified according to the language used in each country. This perspective allows us to further discuss some cultural aspects.

In the Japanese national curriculum, which is called the 'Course of Study', the introduction of proof takes place in grade 8, and the textbooks usually include chapters on proof in geometry. One characteristic of the statements in Japanese textbooks is that all statements required to be proven are about general objects, and statements about specific objects are not for proving (Miyakawa, 2017). This means that the statement to be proven is considered as a universal proposition rather than as another type of proposition, such as a singular or existential proposition. Regarding the treatment of universally valid statements, some previous studies have pointed out the problem of the relationship between natural and mathematical language from a logical point of view (e.g. Cabassut et al., 2012; Durand-Guerrier et al., 2012). We think that it is also important to discuss this matter by focusing on the ordinary language used in a given country, because this may affect how students engage with mathematical proof.

In the case of the universal proposition, in many countries the statement is usually formulated as a phrase like "The sum of the interior angles of *any* triangle is 180°" or "The sum of *any* two even integers is even". While these statements are also objects for proving in Japan, universal quantification using ordinary language and words such as 'any' and 'all' is rarely encountered in secondary mathematics textbooks; that is to say, universality is rarely formulated in written form. Often, Japanese students (and teachers as well) are required to interpret a mathematical statement to be proven as a universal proposition without any quantification. This is also the case for statements in other domains aside from geometry. For example, in lower secondary school algebra universal propositions concerning numerical properties, such as "The sum of two even numbers is an even number", are formulated and proven without a quantifier. This is probably due to the fact that the Japanese language does not use articles [3]. In addition, in Japanese grammar, there is no distinction between the singular and plural form of a noun in both writing and speaking. This matter may also influence understanding of the words related to quantification. However, we would not like the reader to misinterpret this as meaning that the Japanese language does not have the words to express quantification. In fact, there are ordinary Japanese words that correspond to English words such as 'any', 'arbitrary', and 'all'. These Japanese words might be orally used by teachers in the classroom. However, these words are used only if the quantification is consciously intended in the proving process. Therefore, most Japanese students have few opportunities to think about quantification in their own language. Most of them do not encounter quantifiers formulated in mathematical words or symbols such as '$\forall$' or '$\exists$', based on predicate logic until they reach the undergraduate level of university.

The same statement could be formulated in different ways, such as in diagrams, ordinary language, and mathematical language. As far as the linguistic formulation of universal quantification is concerned, there is no explicit progression from implicitly quantified statements through verbally quantified statements to formally quantified statements in the Japanese curriculum. It may be a matter of the semantic contamination between natural language and mathematical discourse that may affect how teachers and students engage with proof and proving of a general statement (see Mejia-Ramos & Inglis, 2011). A previous study targeting prospective Japanese primary teachers found that most of the participants could not evaluate the validity of solutions of a proving task in terms of a universal aspect of the statement (Shinno *et al,* 2012). This is probably because most prospective primary teachers, who were not mathematics majors at university, do not have opportunities to learn how to use universal quantification in their proving activities. Of course, we cannot conclude that Japanese students' or teachers' difficulties concerning universal quantification are due solely to linguistic issues. Rather, what we would like to suggest here is that there is a gap between the levels of linguistic formulation of statements with regard to the treatment of the universal quantifier in the Japanese curriculum. It seems that this gap may affect the teaching and learning of mathematical proof across the grades.

An exceptional case: proof by mathematical induction

It is important to note that there is a remarkable exception to this point. A statement to be proven by mathematical induction is a universal proposition explicitly described as 'for *all* natural numbers' using ordinary Japanese language. This linguistic issue is also related to the epistemological nature of the Mathematical Theorem. We will illustrate this by sharing a typical example from a Japanese textbook. Our focus is not, however, on the details, but rather on what the example reveals about how a statement using ordinary words with an explicit universal quantification may matter, as well as the epistemological nature of the Mathematical Theorem.

In Japan, proof by mathematical induction is taught in the chapter 'Sequences'. This is official content that is included in the upper secondary school subject 'Mathematics B' which is mostly studied in grade 11. Figure 1 and Figure 2 are excerpted from this chapter (Takahashi *et al.,* 2012). Figure 2, which includes a proof by mathematical induction, is introduced in the final part of the chapter. One can find similar descriptions elsewhere, for instance, in Harel (2002), Polya (1954), Tall *et al.* (2012), and probably in many textbooks in other countries.

Figure 1 shows a description of the introduction of the formula:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

There is a statement ($1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1) / 6$) and its proof, where a ready-made identity ($k^3 - (k-1)^3 = 3k^2 - 3k + 1$) is applied for deducing the original statement. This case is similar to what Tall *et al.* (2012) illustrate as an algebraic proof. On the other hand, one can notice that the equations, including the conventional notation '…', are used for representing 'the sum of the first *n* terms' and the operations. This notation in the proof involves some ambiguities from the mathematical point of view as the operations related to the dot notation, such as 'adding to each side of the equation' or transforming these equations into $1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1) / 6$, are carried out without any explanation. Additionally, one may also notice that universal quantification is not used for the formulations of the statement or proof in Figure 1.

In Figure 2, unlike in Figure 1, there is the statement which includes a universal quantification formulated by using the ordinary word 'all'. As we have already mentioned, since universal quantification is rarely used in Japanese textbooks, even in natural language, this explicit use is an exceptional case. In Figure 2, the statement is proven by mathematical induction, although the reference to the Principle of Mathematical Induction (*i.e.,* Peano's fifth axiom for the foundation of natural numbers) is still implicit. In Figure 2, the notation '…' is also used in a similar manner as in Figure 1, but it is only used to represent the equation rather than to prove the statement or in operations on equations. The conventional use of dot notation can become more rigorous if predicate logic and proof by mathematical induction are introduced. When comparing Figure 1 with Figure 2, one can identify an essential transition between them, which is related to the nature of theories in which the statement and proof take place. We will elaborate this transition as a 'reference epistemological model' in the next section.

The necessity of a reference epistemological model

In the previous sections, we discussed some linguistic issues related to proof that can be found in the Japanese curriculum. Although the illustrations we made are taken from the Japanese context, it is important to mention the theoretical and methodological implications of the present study in order to extend the discussion from the local context to the international context. Otherwise, it is difficult to exchange our research results within the international community of mathematics education. For this reason, we will pay special attention to epistemological issues related to proof. The epistemological perspective is crucial for any researcher to

Statement: $1^2+2^2+3^2+\ldots+n^2=n(n+1)(2n+1)/6$

Proof: Consider a given identity $(k+1)^3-k^3=3k^2+3k+1$,

For $k=1$, $2^3-1^3=3\cdot 1^2+3\cdot 1+1$

For $k=2$, $3^3-2^3=3\cdot 2^2+3\cdot 2+1$

For $k=3$, $4^3-3^3=3\cdot 3^2+3\cdot 3+1$

…………………………

For $k=n$, $(n+1)^3-n^3=3\cdot n^2+3\cdot n+1$

By adding each side of the equations,

$(n+1)^3-1^3=3(1^2+2^2+3^2+\ldots\ldots+n^2)+3(1+2+3+\ldots\ldots+n)+n$

Hence, $3(1^2+2^2+3^2+\ldots\ldots+n^2)=(n+1)^3-1^3-3(1+2+3+\ldots\ldots+n)-n$

$=(n+1)^3-3\cdot\{n(n+1)/2\}-(n+1)$

$=n(n+1)(2n+1)/2$

Therefore, $1^2+2^2+3^2+\ldots+n^2=n(n+1)(2n+1)/6$

Figure 1. An excerpt from the chapter 'Sequence' in the textbook. [4]

Statement: The following equation holds for all natural numbers.

$1+3+5+7+\ldots+(2n-1)=n^2$ (☆)

Proof: Show (☆) holds for $n=1$.

On the left side of the equation, the only term is 1,

and on the right side of the equation, $1^2=1$.

Thus, it has been shown that (☆) holds for $n=1$.

Show that if (☆) holds for $n=k$, then it also holds for $n=k+1$.

Assume (☆) holds for $n=k$, that is,

$1+3+5+7+\ldots+(2k-1)=k^2$.

Add $2k+1$ to both sides of the equation. It must then be shown as follows.

$1+3+5+7+\ldots+(2k-1)+(2k+1)=k^2+(2k+1)=(k+1)^2$

The above equation shows that (☆) holds for $n=k+1$.

Since both the basis and inductive step have been performed,

by mathematical induction, (☆) holds for all natural numbers.

Figure 2. Proof by mathematical induction adapted from the textbook.[5]

understand the object of the study undertaken (Balacheff, 2008; Reid, 2015). This perspective may correspond to what Bosch and Gascón (2006, 2014) called the 'reference epistemological model' (REM), which constitutes the basic theoretical lens for researchers to analyze differing mathematical knowledge among different institutions (Chevallard, 1985/1991). Such an REM is also needed for the development of curricula because this model can help to clarify what constitutes proof in the curriculum of a given country.

There is another purpose of an REM. It is to detach ourselves, as researchers, from educational institutions where the practices of mathematics education occur, such as the development of curricula, textbooks, teaching materials, *etc*. Japanese researchers, however, cannot easily detach themselves from educational institutions. This is probably due to the university researcher's close relationship to the community of mathematics education; for example, he or she may be a textbook author, a member of the advisory board of the Ministry of Education, a teacher educator or an adviser to in-service teachers in schools. In fact, research on mathematics education in Japan often aims at developing a curriculum that may have some impact on the national curriculum or mathematics textbooks. The theoretical and methodological reflection that we will address is important because, as Bosch and Gascón (2006) stated, "we need to elaborate our own 'reference' model of the corresponding body of mathematical knowledge" (p. 57). To conceptualize such an REM we need to be able to free ourselves from our educational institution. We think that this detachment is a crucial point, especially for Japan.

A proposition for an REM

Figure 3 represents the basic tenets of the REM we have developed, based on some prior research results in the international context: the triplet which constitutes the Mathematical Theorem and some additional input. In Figure 3, there are three layers according to the levels of theory, which are adapted from the distinction between 'small theory' and 'large theory' (Hanna & Jahnke, 2002) and the idea of local organization (Freudenthal, 1971, 1973). Statements and proofs are interrelated and take place at each theoretical level, and the continuous evolution of each element may depend on mathematical domains wherein the proof is carried out. We call these layers, respectively, 'real-world logic', 'local theory', and 'axiomatic theory'. The first level is not the primary focus of study in secondary mathematics, while it may be involved in the process of teaching mathematical proof. When someone accepts a geometric property to be true by means of a physical experiment or a measurement of a real-world object, it can be interpreted that the theory behind this argument is 'real-world logic'.

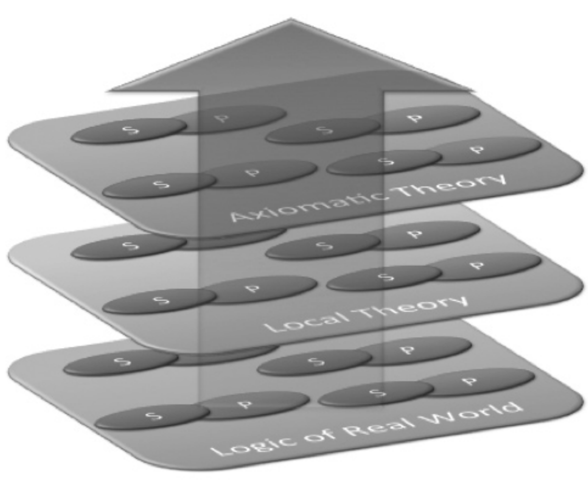

Figure 3. *An image of the proposed REM.*

We suggest that the developments of each element are usually implicit but essentially determine the teaching and learning of mathematical proof. Even though mathematical proof based on the axiomatic theory level is not explicitly dealt with, its implicit nature might be involved in the mathematical proof of secondary school mathematics. This is what we have already discussed, relating to the distinction between *shōmei* and *ronshō* described at the beginning of the paper. For elaborating the two levels of the nature of theories it is reasonable to call the two distinct levels of the nature of theories 'the local theory' and 'the quasi-axiomatic theory' respectively. We added the prefix 'quasi' here because the axiomatization in secondary schools is not strict. The terms axiom or postulate are not used, some properties are introduced after observation, and some are admitted implicitly. Therefore, the transition between local and quasi-axiomatic theory can be essential in developing in developing a curriculum in Japan. In this transition, the development in other elements, statements and proofs are also crucial for the (local or axiomatic) nature of theory.

The two levels, local and quasi-axiomatic theory, may allow us to understand the transition within the textbook chapter which includes proof by mathematical induction (*e.g.,* Figure 1 and 2). In doing so, we can become aware that the teaching of proof by mathematical induction can be seen as 'an exceptional case' in the Japanese curriculum, because one can find a statement with an explicit universal quantification using ordinary words as well as the evolution from local theory to quasi-axiomatic theory. By showing such an example, we intend to illustrate how the proposed REM may work to describe and analyze the evolution that can be identified in the curricula of different countries.

Final remarks

Since each country has its own curriculum, we cannot ignore the influences of curricular contents and sequencing on students' constructions of mathematical proof (Hoyles, 1997). In this paper, we illustrated how cultural aspects (linguistic issues in particular) may potentially affect curriculum development of mathematical proof in the Japanese educational context. As a result, we discussed how the Japanese language may influence the formulation of universal quantification as well as different technical meanings of proof in the Japanese language by comparing with other languages such as English and French. This issue is a challenge in curriculum development in that there may be obstacles in applying prior international research results to the Japanese context. This also implies that it is important to consider cultural differences when examining mathematical proof in different countries.

On the other hand, we also mentioned the necessity of an REM to extend the discussion from the local context to the international context. Regarding the proposed REM, we conceptualized that the statements and proofs are interrelated according to the nature of theory and illustrated how this model helps to describe the evolution of each element. This model may help us to understand the gaps in the evolution of the formulation of a statement and how different meanings of proof relate to the distinction between local and (quasi-) axiomatic levels of theory. Consequently, it is meaningful for Japanese researchers (and hopefully for international researchers as well) to understand why teaching of proof by mathematical induction is 'an exceptional case' in the Japanese curriculum.

Further research is needed to examine how the linguistic issues we identified influence actual classroom lessons and how the constructed REM can analyze such classroom phenomena. Although we have focused on specific linguistic issues related to proof, it is also important to investigate different cultural constraints in the educational contexts of Japan and other countries. Since what constitutes 'proof' varies from institution to institution and country to country, further international comparative studies are needed to develop a deeper understanding of each country's curriculum and classroom teaching. The model we have outlined may be elaborated upon by future empirical studies.

Notes

[1] It should be noted that these three characters have several other meanings, since the same character can have both different meanings and different pronunciations.

[2] In fact, at the most recent conference held in 2016, there were eleven presentations within the category *shōmei* (including *ronshō* and *setsumei*). Four of those were categorized as *shōmei*, two studies were categorized as *ronshō*, three studies were categorized as *setsumei* (explanation) and two studies were on other topics (logical thinking and justification).

[3] We do not intend to claim that this feature of the Japanese language necessarily affects how universal quantifications are spoken and written. A reviewer suggested to us that the Russian language also does not use articles, but many Russian textbooks do use universal predicates. In advanced-level Japanese textbooks, there are many universal quantifiers based on predicate logic. What we would like to point out here is the fact that Japanese secondary textbooks rarely use universal quantification formulated in ordinary Japanese language.

[4] The original of Figure 1 can be found in a Japanese mathematics textbook (Takahashi, 2012, p. 23). The statement and proof were translated by the authors.

[5] The original of Figure 2 can be found in a Japanese mathematics textbook (Takahashi, 2012, p. 40). The statement and proof were translated and rearranged by the authors. Although there are no essential modifications, the words 'base step' and 'inductive step' have been added to the proof text.

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